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Vortex dynamics of viscous fluid flows. Part 1. Two-dimensional flows

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The method of product integration is applied to the vortex dynamics of two-dimensional incompressible viscous media. In the cases of both unbounded and bounded flows under the no-slip boundary condition, the analytic solutions of the Cauchy problem are obtained for the Helmholtz equation in the form of linear and nonlinear product integrals. The application of product integrals allows the generalization in a natural way of the vortex dynamics concept to the case of viscous flows. However, this new approach requires the reconsideration of some traditional notions of vortex dynamics. Two lengthscales are introduced in the form of a micro- and a macro-scale. Elementary 'vortex objects' are defined as two types of singular vortex filaments with equal but opposite intensities. The vorticity is considered as the macro-value proportional to the concentration of elementary vortex filaments inhabiting the micro-level. The vortex motion of a viscous medium is represented as the stochastic motion of an infinite set of elementary vortex filaments on the micro-level governed by the stochastic differential equations, where the stochastic velocity component of every filament simulates the viscous diffusion of vorticity, and the regular component is the macro-value induced according to the Biot–Savart law and simulates the convective transfer of vorticity.

In flows with boundaries, the production of elementary vortex filaments at the boundary is introduced to satisfy the no-slip condition. This phenomenon is described by the application of the generalized Markov processes theory. The integral equation for the production intensity of elementary vortex filaments is derived and solved using the no-slip condition reformulated in terms of vorticity. Additional conditions on this intensity are determined to avoid the many-valuedness of the pressure in a multi-connected flow domain. This intensity depends on the vorticity in the flow and the boundary velocity at every time instant, together with boundary acceleration.

As a result, the successive and accurate application of the product-integral method allows the study of vortex dynamics in a viscous fluid according to the concepts of Helmholtz and Kelvin.

1. Introduction

The foundations of the study of vortex motion in fluid flows were laid by Helmholtz (1858) and Lord Kelvin (1869). Their fundamental works were the origin of the vortex dynamics concept based on the physical representation of the fluid mechanics as vortex motion that determines all the physical characteristics of a flow. The rationality of this representation is completely demonstrated in models of perfect incompressible flows, where the vorticity evolution can be determined from the vorticity field itself in accordance with the Helmholtz and Kelvin theorems, and velocity and pressure fields are found from the vortex motion. The description of the fluid mechanics in the framework of the vortex dynamics concept has a number of fundamental advantages as compared with the traditional description in velocity–pressure terms. First, the most fruitful ideas and theories in aerodynamics are mainly connected with the physical pattern of vortex motions. Therefore, the study of vortex field structures in real flows leads to a deeper comprehension of the nature of the physical phenomena, and may even, sometimes, lead to an explanation of them. This is convincingly demonstrated in the remarkable work of Batchelor (1973). Secondly, the vorticity distributions in flows of practical interest is often compact, whereas the velocity and pressure fields extend everywhere. Therefore, the flow description in vorticity terms requires less information, making vortex dynamics methods more effective. Finally, these vortex methods are more convenient and easy to use as they are based upon the Lagrangian representation of media motion. Modern computers allow these advantages of vortex techniques to be used effectively. It should be especially noted that the vortex method is one of the most effective computational techniques for fluid flow simulation.

Vortex dynamics theory is a rapidly developing branch of fluid mechanics. Numerous analytic and numerical methods have now been created within the framework of this theory to calculate vortex motions in both inviscid and viscous media. A complete survey of the works in the field is outside the scope of this paper, and we merely indicate here the excellent surveys of Aref *et al.* (1988, 1989), Beale & Majda (1984), Hald (1991), Leonard (1980, 1985), Puckett (1991), Sarpkaya (1989) and Sethian (1991), where different approaches to vortex dynamics are analysed, and a vast bibliography is given.

In spite of the considerable achievements in the field, there are a number of fundamental problems in the vortex dynamics of viscous flows which have not yet been satisfactorily solved. One fundamental problem concerns the generalization of the vortex dynamics concept of ideal flows to the viscous case. This problem arises because the vorticity diffusion and the boundary vorticity production violate the Helmholtz and Kelvin theorems and, therefore, the principal basis of the vortex dynamics concept vanishes. As a result, the problem arises of representing vorticity convection, diffusion and boundary production processes in a viscous fluid in terms of the motion of an infinite set of some 'vortex objects'.

There are both traditional analytical methods of mathematical physics and numerical methods of solving the Helmholtz equation, such as finite and boundary element techniques and finite difference methods (see Norrie & de Vries 1978; Thompson & Wu 1973) that do not require describing viscous fluid motion in the framework of the vortex dynamic concept. In these methods, the Lagrangian approach is not used; here, the vorticity is not represented in the form of a set of vortex objects, but rather is considered as some unknown variable which has to be determined from the Helmholtz equation. Note, however, that the traditional methods of mathematical physics unfortunately permit the solution of the problem only when the flow studied has high degree of the symmetry such that separation of variables is possible in the equation. Besides, none of these methods of solving the Helmholtz equation in vorticity terms has any obvious advantages over analogous techniques where the Navier–Stokes equations are solved in the velocity–pressure variables. Moreover, there are some difficulties in formulating the boundary condition on the vorticity magnitude in viscous flows. Therefore, additional problems arise in the application of such approaches in satisfying the no-slip condition in the vorticity terms.

In the past two decades many different numerical methods have been proposed for computing the vortex motion of an incompressible viscous fluid in the limits of the vortex dynamics concept. Among those there are, however, methods which do not give the exact problem solution, as the viscosity effect is taken into consideration only approximately, for instance Ashurst's (1979) technique, where the diffusion is taken into account by exponential spreading in time of vortex cores. This method has limited application (see Greengard 1985).

Most progress in the solving of this problem was achieved in the computational technique of random walks based on Chorin's works (1973, 1978, 1980, 1982). The fundamental physical idea of this technique consists in the vorticity diffusion process being simulated by random walks added to the usual inviscid regular motion of vortices. Marchioro & Pulvirenty (1982) first gave the correct theoretical basis for applying the random walks to simulating solutions for the vortex dynamics equations in a viscous fluid. First, they showed that the quasi-linear Helmholtz equation can be interpreted as the forward Kolmogorov equation for some stochastic process of vortex blob motion which is constructed by the standard procedure of successive approximations as the limit of processes involving the linearized Helmholtz equation. Secondly, using the 'propagation of chaos' representation (see McKean 1969), the authors showed that a solution of the Helmholtz equation can be represented through the stochastic motions of an infinite set of vortex blobs. A more detailed description of the random walks method including its foundations and a vast bibliography is contained in Puckett (1991).

Nevertheless, the random walks method has some fundamental shortcomings. Milinazzo & Saffman (1977) showed that this method requires a considerable number of vortices, $N \sim Re$, to simulate viscous flows correctly. Besides, there are some difficulties in satisfying the no-slip boundary condition. For example, the boundary-layer equations, which are not valid in the vicinity of a flow separation, have additionally to be solved to describe vortex production on boundaries. A more detailed criticism of this method can be found in Sarpkaya (1989). Hence, we can say with confidence that the analytic theory of the vortex dynamics of a viscous fluid is not yet formulated. The existence of such a theory would allow well grounded and effective calculation techniques to be created.

Meanwhile, the method of product integration has been widely developed in different domains of physics. This method is mainly used to study physical phenomena governed by partial differential equations of parabolic type. The path integral (product integral) was first introduced by Wiener in 1923 to solve linear Brownian motion problems. Later, Feynman introduced the continual integral notion, a complex version of the path integral, to solve Schrödinger's linear equation (see Feynman & Hibbs 1965). In recent years this method has been generalized to the solution of quasi-linear equations. The nonlinear version of the product integral was introduced by Maslov (1976) to solve the nonlinear Schrödinger equation, and the asymptotic technique for this integral evaluation was also worked out by him. Product integration is now a branch of mathematics that has found principal application in quantum theory, statistical physics, and the examination of wave propagation in random media, and is spreading to other fields of physics. Asymptotic and numerical methods have been worked out to evaluate these integrals. Unfortunately, this method has not been applied to fluid mechanics, apart from the sole case of the application of product integrals to solve the Hopf functional equation in turbulence theory. A complete account and vast bibliography on this subject can be found in the monograph of Monin & Yaglom (1975).

The purpose of the present work is the construction of the analytic vortex dynamics theory of an incompressible viscous fluid on the basis of the product-integral method.

We intend to obtain the analytic solution of the Cauchy problem for the Helmholtz equation satisfying the no-slip condition on the boundaries in the form of a nonlinear product integral. The possibility of applying this approach arises from the parabolicity of the Helmholtz equation in a viscous fluid. The harmonic connection between product integration and continual Markov processes (the product integral expresses the mean value of a functional over some Markov process) allows these examinations to be carried out in the framework of the vortex dynamics concept. Moreover, one can follow the traditional vortex dynamics formulation: a vorticity field is represented in the form of a set of some vortex objects, and their laws of motion are assigned so that the dynamics of the vortices satisfies the Helmholtz equation. From our point of view such an approach is well grounded and clear from a physical standpoint.

In this addition, the product-integral method requires the motion of the vortex objects be guided by some Markov process. One can perceive here that some constructions in the method developed are analogous to those of Chorin. Indeed, our study uses some ideas from the numerical random walks method. However, there are some principal differences between these two methods because the application of the product-integral method to vortex dynamics requires reconsideration of the notion of stochastic vortex objects and ways of formulating their motion laws. Also these examinations will use some recent results in product integration theory, in particular the new way of satisfying the boundary conditions. Therefore, we give in §2 a general introduction to the application of product integration and the necessary bibliography, discussing the specific problems arising in solving the problem formulated here.

In §3 a concrete model of the vortex dynamics of two-dimensional fluid flows is constructed on the base of the principles formulated in §2. Using this model, the Cauchy problem for the Helmholtz equation is solved, first in §4 for the case of unbounded flows using both linear and nonlinear forms of product integrals, and second in §5 for flows with arbitrary moving boundaries under the no-slip condition in the form of a nonlinear product integral.

This work generalizes the authors' previous work (Ostrikov & Zhmulin 1991).

2. The application of the product-integral method to viscous vortex dynamics: reasons, reconsideration of concept and physical pattern

This Section is devoted to some general questions. First, we shall outline the application of the product-integral method (PIM) to vortex dynamics in a viscous fluid and show that it is the most natural means of generalizing the vortex dynamics concept of idea! fluids to viscous fluids. Secondly, the general theses of PIM are formulated and the specific problems that occur in the application of PIM to viscous vortex dynamics are discussed. Overall, this Section gives a basis for revising some fundamental notions in viscous vortex dynamics and outlines ways of solving a problem by using PIM. We consider here both two- and three-dimensional flows, though the practical PIM application to the three-dimensional case will be done in a future paper (Part 2).

The evolution of vorticity in an incompressible viscous fluid is governed by

$$\frac{\partial \boldsymbol{\Omega}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{\Omega}-(\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{u}=\nu\Delta\boldsymbol{\Omega}, \tag{2.1}$$

$$\nabla\cdot\boldsymbol{u}=0, \tag{2.2}$$

$$\boldsymbol{\Omega}=\nabla\times\boldsymbol{u}, \tag{2.3}$$

$$\boldsymbol{u}|_S=\boldsymbol{v}_s, \tag{2.4}$$

where (2.4) is the no-slip condition on the boundaries moving with a given velocity $\boldsymbol{v}_s$.

Equations (2.2) and (2.3) allow the velocity field of a flow to be determined from the vorticity through the Biot–Savart law

$$\boldsymbol{u}(\boldsymbol{x},t)=\frac{1}{4\pi}\int\frac{\boldsymbol{\Omega}(\boldsymbol{y},t)\times(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}\mathrm{d}y+\boldsymbol{\nabla}\phi(\boldsymbol{x},t). \tag{2.5}$$

Here, the harmonic function, ϕ determines a potential flow added to satisfy the boundary condition and can be expressed through the vorticity in a flow. After substituting (2.5) into (2.1) the Helmholtz equation becomes a quasi-linear one of parabolic type, related to the vorticity only.

The initial Cauchy problem is formulated for this equation; the vorticity evolution $\boldsymbol{\Omega}(\boldsymbol{x},t)$ is to be determined when the initial vorticity $\boldsymbol{\Omega}_0(\boldsymbol{x})$ is given at the time t_0.

2.1. *Characteristics of the vortex dynamics concept in a viscous fluid*

In an inviscid flow, when $\nu=0$, the right-hand side of (2.1) vanishes, and the vorticity evolution satisfies the Helmholtz and Kelvin theorems: vortex lines are material ones and the intensity of vortex tubes is maintained in time. The vortex dynamics of an ideal fluid is based on these theorems and can be described as the evolution of some deterministic objects in the form of vortices. This representation of ideal fluid flow is called the vortex dynamics concept. To apply this concept, two correlated problems have to be solved:

(i) to represent the vorticity field as a set of vortex objects, each of which induces a velocity field according to the Biot–Savart law,

(ii) to define the laws of motion of the vortex objects in the form of a set of differential equations.

In ideal-fluid models the vorticity field is usually represented in the form of a set of δ-shape vortices, such as vortex filaments, vortons, vortex dipoles or regularized vortices in the form of thin vortex tubes or vortex blobs. The derivation of differential equations describing the motion of such vortex objects is based on the property of vortex lines being liquid, and the motion of every vortex object is, therefore, considered to be caused by the velocity induced by the rest of the vortices according to the Biot–Savart law. In the three-dimensional case, additional equations are constructed for the intensity of δ-shaped vortices or vortex blobs to describe the stretching effect of vortex tubes.

Unlike inviscid models, where only the convective transfer of vortices is considered, in viscous flows there are three correlated processes: convection, diffusion and the production of the vorticity. Vorticity diffusion process arises from molecular transfer between liquid particles. The vortex production process arises from the no-slip condition (2.4) on boundaries and can be explained as follows. If in an inviscid fluid the potential ϕ in (2.5) is determined under the no-flow boundary condition, $(\boldsymbol{u}-\boldsymbol{v}_s)\cdot\boldsymbol{n}|_S=0$, for any vorticity in a flow, then in a viscous fluid this potential cannot be found for arbitrary vorticity to satisfy (2.4). Therefore, in the viscous case, some vorticity quantity must be produced on the boundaries at every instant of time in order for this potential to be determined.

The appearance of these additional processes in a viscous flow creates the principal difficulties in the application the vortex dynamics concept. Owing to vorticity diffusion and boundary production, the Helmholtz and Kelvin theorems are not applicable and so the vortex lines are not material ones and the vortex tube intensities are not maintained. Nevertheless, vorticity production can be described as a production process of deterministic vortex objects by analogy, for instance, with the simulation of separated ideal flows. On the other hand, the diffusion process cannot be described by the techniques used in inviscid flows. Indeed, when a local vortex arises in a viscous flow as a hydrodynamic object, it diffuses immediately, producing a vorticity field in the whole flow field. Therefore, it is difficult to represent vortex motion in the form of movements of deterministic vortex objects guided by some differential equations.

To overcome these difficulties and contradictions, the mathematical model has to permit the diffusion to be described in terms of the motion of some deterministic objects, with an adequate description of the convection and production phenomena. So far, the theory of continuous Markov processes with PIM, correlated with this theory, is the only way to solve this problem.

2.2. *The product-integral method as a way of describing diffusion phenomena*

The continuous Markov processes theory has been developed from works on Brownian motion (see Einstein 1956). These works gave rise to the method in which diffusion phenomena are studied by the description and examination of diffusing particle motions. This method has been successfully applied to many different domains of physics (see Van Kampen 1984).

From the mathematical standpoint the harmonic connection between continuous Markov processes and solutions of linear partial equations of parabolic type was discovered by Kolmogorov (1931). Later, this result was generalized on the case of quasi-linear equations of this type (see Freidlin 1967). Wiener (1923) introduced the path integral (product integral) as one over the functional space containing all possible paths of some continuous Markov process, and this was the origin of the effective analytic method of examination of both linear and quasi-linear parabolic equations. The main information on the product integral theory can be obtained in the works of Albeverio & Hoegh-Krohn (1976), Daletsky (1962), Dynkin (1965), Egorov, Sobolevsky & Yanovich (1983), Feynman & Hibbs (1965), Maslov (1976) and Freidlin (1985).

To solve the Cauchy problem analytically for any equation of parabolic type in the framework of this theory, the following steps are used (see figure 1). First, the continuous Markov process is constructed using stochastic differential equations and the introduction of the production and disappearance of stochastic objects. The value of this construction is in the possibility of interpreting the initial partial equation as a forward Kolmogorov's one. Finally, the Cauchy problem is solved for the initial equation by calculating the mean value of the special functional over the constructed Markov process in the form of the product integral.

The main physical characteristic of the diffusion process is here the transition probability density $p(x,t,y,\tau)$ defined as the probability that diffusion objects move to the position x at time t, if they start out from position y at time τ. The problem of the analytic determination of this probability density is solved by PIM. The product integral is here interpreted as the sum of contributions to $p(x,t,y,\tau)$ from all possible trajectories of diffusion objects moving to point x from y during the time interval $[\tau,t]$.

Particular attention should be paid to the peculiarity of the physics of the diffusion phenomena, which arises because the forward Kolmogorov equation is written with respect to functions which are usually, in some sense, proportional to the concentrations of diffusing objects. For example, the Brownian diffusion equation describes the concentration directly. In the heat-transfer process the temperature is determined through the Boltzman function which determines the concentration of the molecules. Moreover, there are two characteristic lengthscales in diffusion phenomena: the motion of diffusing objects occurs, as a rule, on the micro-level, while the diffusion

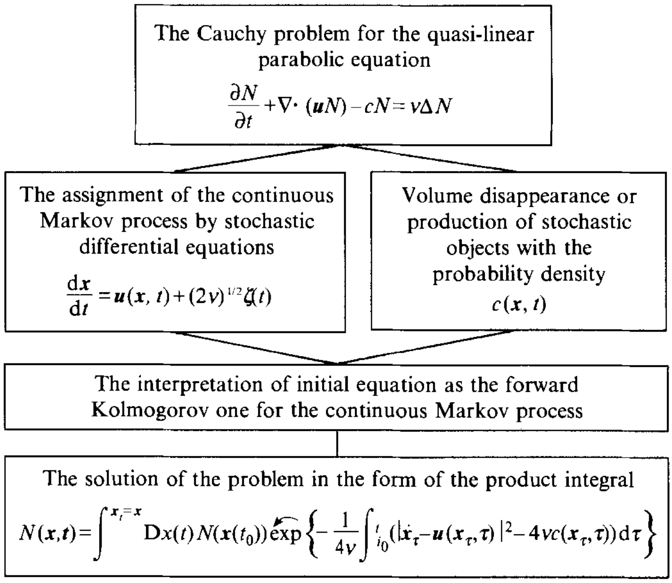


FIGURE 1. A schematic of the Cauchy problem solution for parabolic-type equations, describing some diffusion process $N(\boldsymbol{x}, t)$ by the continuous Markov process theory, where $\boldsymbol{u}(\boldsymbol{x}, t)$ is the convective velocity of diffusing objects motion, $c(\boldsymbol{x}, t) > 0$ is the probability density of diffusing objects disappearance, and $c(\boldsymbol{x}, t) < 0$ is that of their production, ν is the diffusion coefficient, and $\zeta(t)$ is white noises. The terms $\boldsymbol{u}(\boldsymbol{x}, t)$ and $c(\boldsymbol{x}, t)$ depend on $N(\boldsymbol{x}, t)$ for the quasi-linear case.

process itself is manifested on the macro-level. Therefore, the convective velocity and the probability density of the production or disappearance of diffusing objects are macro-level characteristics in the Kolmogorov equation. This fundamentally influences the construction of stochastic equations, their physical interpretation and the interpretation of solutions written in the form of product integrals.

The appearance of boundaries in the diffusion process can lead to additional physical phenomena, such as absorption, production, reflection and passing through a boundary of diffusing objects. These phenomena change the transition probability density of the continuous Markov process. In order to apply PIM effectively for the description of such boundary phenomena, it is convenient to apply the theory of generalized Markov processes developed by Portenko (1982). In this method the stochastic process is continued outside the domain studied, and the Kolmogorov equation is different to that drawn on figure 1 in the ϵ-neighbourhood of the boundaries as follows:

$$\frac{\partial N}{\partial t}+\nabla\cdot((\boldsymbol{u}+\boldsymbol{u}_\epsilon)N)-(c+c_\epsilon)N=\nu\Delta N+q_\epsilon, \tag{2.6}$$

where $\boldsymbol{u}_\epsilon$ is the additional convective velocity, c_ϵ is the additional probability density of the production or disappearance of objects, and q_ϵ is the intensity of the objects' ejection. The support of $\boldsymbol{u}_\epsilon$, c_ϵ and q_ϵ is the ϵ-neighbourhood of boundaries. Equation (2.6) can be solved by the standard PIM drawn in the figure 1. Choosing the quantities $\boldsymbol{u}_\epsilon$, c_ϵ and q_ϵ within the ϵ-layer, one can describe a broad class of boundary phenomena. In the limit as $\epsilon\to 0$, the solution of an initial problem can be obtained in the form of the product integral.

The parabolic type of the quasi-linear Helmholtz equation (2.1) suggested to us the idea of applying PIM to describe the vortex dynamics in a viscous fluid. Clearly, the stochastic process, constructed for applying PIM to solving the Helmholtz equation, can be interpreted as the stochastic process of the motion of some vortex objects. Thus, using PIM allows the vortex dynamics concept to be generalized to viscous fluid because the latter can be represented in the form of the motion some deterministic objects guided by some equations. Unlike an ideal fluid, those equations will be stochastic ones determining some continuous Markov process. As a result, the deterministic description of vortex trajectories is unable to describe viscous vortex dynamics within the framework of the vortex dynamics concept. The main characteristic of viscous vortex motion becomes the transition probability density $p(x, t, y, \tau)$, and the principal problem of the viscous model construction consists in the determination of this probability density so that the vorticity $\boldsymbol{\Omega}(\boldsymbol{x}, t)$, arising as a result of the stochastic motion of vortex objects, would satisfy (2.1). Note that this inapplicability of deterministic approach is analogous to that when the transition to quantum from classical mechanics is made in Feynman's interpretation (Feynman & Hibbs 1965). However, there is no more profound analogy here because there is a difference between the nature of stochastical properties of these phenomena.

Note that the numerical random walks method of Chorin, mentioned in §1, is based on this idea. However, the particular version of the stochastic processes, constructed in this method, cannot be applied to PIM for analytical solutions of the problem. The reason for this will be considered in the next Section.

2.3. *The main problems of the application of PIM to viscous vortex dynamics*

In order to apply PIM to viscous models, first the two problems (i) and (ii) given in §2.1 have to be solved to construct the stochastic process and to give its physical interpretation. In this case, the correctness criterion for solving these two problems is whether the Helmholtz equation can be interpreted as the forward Kolmogorov one for the Markov process constructed. The quasi-linearity of the Helmholtz equation causes difficulties in the realization of this programme. It appears that representing the vorticity field in the form of a set of vortex objects, as used in ideal flow models, is not possible in the application of PIM to viscous flow. We intend to clear up this basic problem using the example of two-dimensional flows.

First, consider representing the vortex field in the form of a set of vortex filaments to solve the problem discussed. In order to do this, the whole vorticity field $\Omega(\boldsymbol{x})$ is divided into small cells ΔS_i. Then one vortex filament of intensity $\Gamma_i = \Omega(\boldsymbol{x}_i)\,\Delta S_i$ is placed in the centre of each cell $\boldsymbol{x}_i$, and the vorticity field will approximately be

$$\Omega(\boldsymbol{x}) \approx \sum_i \Gamma_i\, \delta(\boldsymbol{x}-\boldsymbol{x}_i). \tag{2.7}$$

Relation (2.7) becomes precise in the limit $\Delta S_i \rightarrow 0$ and gives a representation of the vorticity field as an infinite set of vortex filaments. The motions of the vortex filaments have to be given in the form of stochastic differential equations which can be reduced to ideal ones, as $\nu = 0$. Therefore, these equations can be written in the form

$$\frac{\mathrm{d}\boldsymbol{x}_i}{\mathrm{d}t} = \sum_{j \neq i} \frac{\Gamma_j}{2\pi} \frac{\boldsymbol{e}_z \times (\boldsymbol{x}_i - \boldsymbol{x}_j)}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^2} + (2\nu)^{1/2} \zeta_i(t), \tag{2.8}$$

where the $\zeta_i(t)$ are independent white noise so that the

$$\int_0^t \zeta_i(\tau)\,\mathrm{d}\tau$$

are independent Gaussian random values with zero mean and variance $2\nu t$. In this case, the vorticity field in a viscous fluid is determined by averaging (2.7) over all realizations of (2.8) as

$$\Omega(\boldsymbol{x}, t) = E[\sum_i \Gamma_i \, \delta(\boldsymbol{x}-\boldsymbol{x}_i(t))], \tag{2.9}$$

where E is the mean value. If we use PIM to calculate this mathematical expectation, we then obtain a further equation for $\Omega(\boldsymbol{x}, t)$:

$$\frac{\partial \Omega(\boldsymbol{x}, t)}{\partial t} - \nu \Delta \, \Omega(\boldsymbol{x}, t) = -\nabla_x \cdot E[\sum_i \Gamma_i \boldsymbol{u}_i(\boldsymbol{x}) \, \delta(\boldsymbol{x}-\boldsymbol{x}_i(t))]. \tag{2.10}$$

Here, the velocity $\boldsymbol{u}_i(\boldsymbol{x})$ is given by the first term on the right-hand side of (2.8) in which $\boldsymbol{x}$ replaces $\boldsymbol{x}_i$.

It is seen from (2.10) that the convective term on the right-hand side does not have the form of one in the Helmholtz equation (2.1) because the velocity $\boldsymbol{u}_i(\boldsymbol{x})$ depends on the coordinates of all vortex filaments and cannot be independent of the sign of the mean value. Thus, this approach fails in the simulation of vorticity convection, though diffusion is described correctly. The main reason for this is due to incorrectly taking into account the quasi-linearity of the Helmholtz equation in the stochastic process construction. Nevertheless, this approach can be applied to simulate flows approximately when the nonlinear interactions between vortices can be neglected. In such flows, the velocity $\boldsymbol{u}_i(\boldsymbol{x})$ does not depend on the vortices coordinates being one of the main flow and can be independent of the sign of the mean value that reduces (2.10) to the Helmholtz equation.

Consider now the vorticity field representation in the form of a set of vortex blobs. It is this representation that is used in Chorin's numerical method and it is standard in inviscid models also. In this technique, the vorticity is represented in the form (2.7) also, but some regularization function $f_\sigma(|\boldsymbol{x}-\boldsymbol{x}_i|)$ replaces the Dirac delta. In the physical sense it means that a vortex of small radius σ is placed at the centre of each cell instead of a point vortex. The continuous Markov process is here constructed analogously to the previous case. Marchioro & Pulvirenty (1982) proved that such a numerical procedure converges to solutions for the Helmholtz equation under the condition $\sigma > h$ only, where h is a characteristic distance between blobs. This condition is necessary in order that a great number of vortex blobs could overlap each other, and the law of great numbers could be applied to calculate the sum on the right-hand side of (2.10), modified for the case of vortex blobs, at every realization of the stochastic process. Esposito & Pulvirenty (1989) generalized this result to the three-dimensional case. However, the condition of the 'overlapping' of vortex blobs contradicts the initial representation of a vorticity field when each vortex cell contains one vortex blob so that 'overlapping' is not considered. Because of this contradiction, Sarpkaya (1989) found this method inadequate in the physical interpretation of the vorticity diffusion process. Indeed, it is difficult to give a physical significance to the parameter σ for 'overlapping' vortex blobs. Furthermore, the existence of this parameter and the function $f_\sigma(|\boldsymbol{x}-\boldsymbol{x}_i|)$ creates fundamental difficulties in the product integral construction. The last circumstance is of most importance for us.

2.4. *Vortex object: reconsideration of the concept*

The discussion in the previous Section shows that the standard representation of the vorticity field as a set of vortex objects must be changed to apply PIM to an analytic description of the vortex dynamics in a viscous fluid. PIM application requires that the

Helmholtz equation (2.1) is to be interpreted as the forward Kolmogorov equation for some Markov process. Therefore, it would be natural to use the physical representation when Kolmogorov's equation describes the magnitude to be proportional to the concentration of diffusing objects, as is usual for the physics of diffusion phenomena (see §2.2). It is just this standpoint that we develop in this work.

Consider now the main physical demands which are to be satisfied in constructing the viscous vortex model. First, the vorticity field must be represented as proportional to the concentration of some elementary vortex objects. In this representation two lengthscales are used: micro- and macro-scales. The physical significance is that elementary vortex objects inhabit the micro-level, while the vorticity field is macro-level magnitude. Note that δ-shaped vortex objects can be used as elementary ones in spite of change of vortex field representation. Secondly, the motion of these vortex objects has to be given by stochastic differential equations of the form (2.8). However, the convective term in these equations must be assigned at the macro-level value. It can be obtained by calculating the main part from the convective velocity in (2.8) with respect to the small parameter of micro-level. As a result, all elementary vortices would have the same convective motion, if they are placed at given macro-point and at some instant of time. This characteristic feature confirms the statistical independence of the motion of different elementary vortices at every small time interval and allows the quasi-linearity of the Helmholtz equation to be accounted for correctly. Third, the volume production or disappearance of elementary vortex objects is possible in the three-dimensional case when the Helmholtz equation has a term $(\boldsymbol{\Omega}\nabla)\boldsymbol{u}$ analogous to the term cN in the equation drawn on figure 1 having a corresponding interpretation. All characteristics, both volume and surface production, of elementary vortex objects must be assigned macro-values also.

Finally, the physical pattern of a viscous fluid flow arises. Elementary vortex objects move stochastically by analogy with the molecules on the micro-level, but a hydrodynamical flow develops on the macro-level as a result of the instantaneous average of the micro-level characteristics. In this connection, the vorticity diffusion phenomenon is the result of random walks of elementary vortices, while convection is their regular displacement against the background of random walks.

This vorticity field representation is radically different to its generally accepted notion in ideal fluid models, and the physical interpretation of an elementary vortex object is the natural problem. Indeed, elementary vortex objects are not the hydrodynamical ones, as is seen from the construction of an elementary vorton which does not have its own flow, but it is their set that characterizes the fluid motion. The authors have assumed from the beginning of this investigation that elementary vortex objects are quasi-particles analogous to those studied in the many-body problem (see Muttuck 1967). More exactly, every elementary vortex object is a quasi-particle appearing as a result of collecting a great number of molecules. However, this quantity must not be too big, otherwise this quasi-particle would become a hydrodynamical object and its motion would not be guided by stochastic laws usually applied to molecular motion. On the other hand, the number of molecules has to be big enough in order that the law of large numbers allows the molecular chaos to be averaged, and the Gaussian law to be applied to the motion of elementary vortex objects. It is seen from this that the physical significance of the micro-scale in the vortex field representation is some intermediate scale between the free path of molecules and the macro-scale, determining the flow. Note that the quasi-particle notion correlates with PIM, as is clearly demonstrated by Muttuck (1967). However, the evidence for this point of view can be confirmed by deeper investigations of kinetic theory, which is outside the scope of this work. Nevertheless, this hypothesis gave us good intuitive grounds for formulating the heuristic assumptions in the mathematical model construction.

3. Two-dimensional viscous fluid flows: principal definitions

The problem of the vorticity evolution will now be examined according to the principles and by the techniques established in §2.

The vortex dynamics of two-dimensional viscous incompressible flows is governed by

$$\frac{\partial \Omega}{\partial t}+(\boldsymbol{u}\cdot\nabla)\Omega = \nu\Delta\Omega, \tag{3.1}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{3.2}$$

$$\Omega \boldsymbol{e}_z = \nabla\times\boldsymbol{u}, \tag{3.3}$$

$$\boldsymbol{u}|_S = \boldsymbol{v}_s. \tag{3.4}$$

The task is to solve Cauchy problem for the Helmholtz equation (3.1), when the vorticity $\Omega_0(\boldsymbol{x})$, compatible with the condition (3.4), is given at initial time t_0.

3.1. *Definition of elementary vortex objects*

The problem is to define a set of simple stochastic vortex objects, the motions of which simulate the solution of equations (3.1)–(3.3). Introduce two types of elementary vortex filaments (EVFs) with intensities $\gamma^+ = \gamma_0$ and $\gamma^- = -\gamma_0$, where γ_0 is small positive number. Every EVF has vorticity of the form

$$\Omega(\boldsymbol{x}) = \gamma^{\pm}\delta(\boldsymbol{x}-\boldsymbol{x}_0)$$

and induces a velocity field

$$\boldsymbol{u}(x) = \frac{\gamma^{\pm}}{2\pi}\frac{\boldsymbol{e}_z\times(\boldsymbol{x}-\boldsymbol{x}_0)}{|\boldsymbol{x}-\boldsymbol{x}_0|^2},$$

where $\boldsymbol{x}_0$ is the EVF's coordinate. Consider that $N = N^+ + N^-$, EVFs of both types are concentrated in some domain B with the lengthscale L, and the distance between EVFs is of order $\epsilon \ll L$. Then $N \sim L^2/\epsilon^2$. Let the domain B be divided into M cells B_α ($B = \bigcup_{\alpha=1}^{M} B_\alpha$) having area ΔS_α and characteristic lengthscale d of order $\epsilon \ll d \ll L$ (see figure 2). We introduce the index sets $J^{\pm}$ of EVF numbers. Denote the numbers of corresponding EVFs placed in cell B_α as $J_\alpha^{\pm}$, and $J_\alpha = J_\alpha^+ \cup J_\alpha^-$. If the number of EVFs of corresponding types in the cell B_α is $N_\alpha^{\pm}$, then $N_\alpha^{\pm} \sim d^2/\epsilon^2$. Also, we assume that $\gamma_0 \sim \Gamma\epsilon^2/L^2$, where Γ is the intensity of the whole vortical domain B. In this case, the total vorticity can be represented in the form

$$\Omega(\boldsymbol{x}) = \sum_{i\in J^+}\gamma^+\delta(\boldsymbol{x}-\boldsymbol{x}_i)+\sum_{i\in J^-}\gamma^-\delta(\boldsymbol{x}-\boldsymbol{x}_i), \tag{3.5}$$

where $\boldsymbol{x}_i$ is the coordinate of ith vortex filament.

Transform (3.5) in such a way as to express the vorticity $\Omega(\boldsymbol{x})$ through the concentration of EVFs and, therefore, write the relation (3.5) in the form of the sum of sums over each cell as

$$\Omega(\boldsymbol{x}) = \sum_{\alpha=1}^{M}\left(\sum_{i\in J_\alpha^+}\gamma^+\delta(\boldsymbol{x}-\boldsymbol{x}_i)+\sum_{i\in J_\alpha^-}\gamma^-\delta(\boldsymbol{x}-\boldsymbol{x}_i)\right). \tag{3.6}$$

The coordinate of the kth EVF can be written as the sum of two terms

$$\boldsymbol{x}_k = \boldsymbol{x}_{\alpha(k)}+\boldsymbol{\xi}_k, \tag{3.7}$$

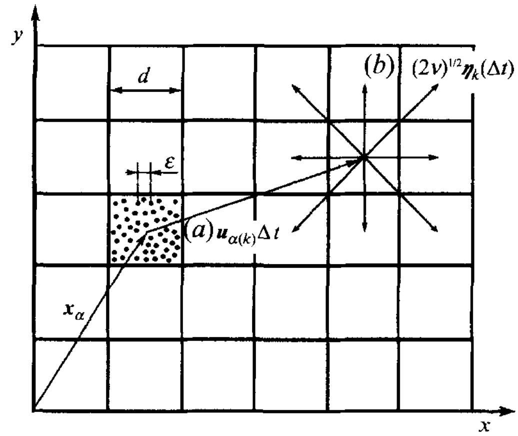


FIGURE 2. Two independent types of EVF motion from cell α at time step Δt: (*a*) the general convective transfer of an EVF by the same velocity $\boldsymbol{u}_{\alpha(k)}$, (*b*) individual random walks $(2\nu)^{1/2}\eta_k(\Delta t)$ of Gaussian form; ϵ is a characteristic distance between EVFs and d is the characteristic lengthscale of a cell.

where $\boldsymbol{x}_{\alpha(k)}$ is the coordinate of the 'centre of gravity' of the cell B_α, in which the kth EVF is placed. The EVF coordinate inside the cell is $|\xi_k| \leqslant d$. As $d \ll L$, we can use the expansion

$$\delta(\boldsymbol{x}-\boldsymbol{x}_k) = \delta(\boldsymbol{x}-\boldsymbol{x}_{\alpha(k)}) + \boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{x}_{\alpha(k)})\,\xi_k + \dots . \tag{3.8}$$

Substituting (3.8) with (3.7) into (3.6), we obtain

$$\Omega(\boldsymbol{x}) = \sum_{\alpha=1}^{M} \gamma_0 (N_\alpha^+ - N_\alpha^-)\,\delta(\boldsymbol{x}-\boldsymbol{x}_\alpha) + O(d). \tag{3.9}$$

Let us introduce the notation $N_\alpha^\pm = N_\pm(\boldsymbol{x}_\alpha)\,\Delta S_\alpha$, where $N_\pm(\boldsymbol{x}_\alpha)$ is the concentration of EVFs of each type in cell B_α. Then, it is obvious, that $N_\pm(\boldsymbol{x}_\alpha) \sim \epsilon^{-2}$. Taking into account this notation, (3.9), in the limit as $\epsilon \to 0$, $d \to 0$, and $\epsilon/d \to 0$, can be written as

$$\Omega(\boldsymbol{x}) = \int_B \gamma_0 (N_+(\boldsymbol{y}) - N_-(\boldsymbol{y}))\,\delta(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \gamma_0 (N_+(\boldsymbol{x}) - N_-(\boldsymbol{x})). \tag{3.10}$$

Note, that $\Omega(\boldsymbol{x}) = O(1)$ due to the definitions of γ_0 and $N_\pm(\boldsymbol{x})$.

Thus, in this case, the EVFs are determined so that the vorticity magnitude is proportional to their concentrations. This definition contains the idea of the existence of two incommensurate lengthscales, namely the micro-scale ϵ characterizing the disposition of EVFs, and the macro-scale L characterizing the vorticity domain. The macro-level's quantitative characteristics are determined in this approach as a result of calculating the principal part with respect to ϵ and d from the corresponding micro-level ones. The velocity field, for example, induced on the micro-level at the point of the kth vortex filament by the set of EVFs, can be determined as

$$\boldsymbol{u}(\boldsymbol{x}_k) = \sum_{\substack{i\in J^+\\ i\neq k}} \frac{\gamma^+}{2\pi}\frac{\boldsymbol{e}_z \times (\boldsymbol{x}_k - \boldsymbol{x}_i)}{|\boldsymbol{x}_k - \boldsymbol{x}_i|^2} + \sum_{\substack{i\in J^-\\ i\neq k}} \frac{\gamma^-}{2\pi}\frac{\boldsymbol{e}_z \times (\boldsymbol{x}_k - \boldsymbol{x}_i)}{|\boldsymbol{x}_k - \boldsymbol{x}_i|^2}. \tag{3.11}$$

When calculations, analogous to those used to obtain (3.10), are made the principal part of (3.11) with respect to ϵ and d will be

$$\boldsymbol{u}(\boldsymbol{x}_k) = \sum_{\substack{\beta=1\\ \beta\neq\alpha(k)}}^{M} \frac{\gamma_0}{2\pi}(N_+(\boldsymbol{x}_\beta) - N_-(\boldsymbol{x}_\beta))\frac{\boldsymbol{e}_z \times (\boldsymbol{x}_{\alpha(k)} - \boldsymbol{x}_\beta)}{|\boldsymbol{x}_{\alpha(k)} - \boldsymbol{x}_\beta|^2}\Delta S_\beta + \boldsymbol{w}(\boldsymbol{x}_k), \tag{3.12}$$

where $|w(x_k)|$ is of $O(d)$ from the magnitude of the first term in (3.12). Under the conditions $\epsilon \to 0$, $d \to 0$ ($\epsilon/d \to 0$), this first term transforms into the well-known Biot–Savart law

$$u(x) = \frac{1}{2\pi}\int \Omega(y)\frac{e_z \times (x-y)}{|x-y|^2}\,\mathrm{d}y. \tag{3.13}$$

3.2. *The laws of vortex filament motion*

The stochastic differential equations of EVF motion can be written in the form

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = \sum_{\substack{\beta=1\\ \beta \neq \alpha(k)}}^{M} \frac{\gamma_0}{2\pi}(N_\beta^+ - N_\beta^-)\frac{e_z \times (x_{\alpha(k)} - x_\beta)}{|x_{\alpha(k)} - x_\beta|^2} + (2\nu)^{1/2}\zeta_k(t). \tag{3.14}$$

Here the number of EVFs of each type $N_\beta^\pm$ in the cell $\mathbf{B}_\beta$ is connected with the coordinates of the EVFs by the correlation

$$N_\beta^\pm(t) = \sum_{i \in J^\pm} \chi_\beta(x_i(t)), \tag{3.15}$$

where $\chi_\beta(x)$ is the characteristic function of the cell B_β, and $x_i(t)$ is the path of the ith vortex filament. The white noises $\zeta_k(t)$ in (3.14) are statistically independent so that the random values

$$\int_0^t \zeta_k(\tau)\,\mathrm{d}\tau$$

are the standard Wiener processes with variance $2\nu t$.

The stochastic differential equations (3.14) determine the continuous Markov process in space R^{2N}. The peculiarity of the motion law (3.14) is that each EVF is moved not by the velocity field (3.11) induced by the rest vortices, but by the principal part of velocity field (3.12) only, that is the macro-velocity. From (3.12) it follows that all EVFs placed in some cell have the same convective velocity apart from their micro-displacements inside each cell, and so the motion of different EVFs is statistically independent in a small time interval.

3.3. *Interpretation of the Helmholtz equation as the forward Kolmogorov equation*

We now demonstrate that the vorticity evolution based on the previous definitions is in agreement with the Helmholtz equation (3.1). We denote

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \quad u = \begin{pmatrix} u_{\alpha(1)} \\ \vdots \\ u_{\alpha(N)} \end{pmatrix}, \quad \zeta_t = \begin{pmatrix} \zeta_1(t) \\ \vdots \\ \zeta_N(t) \end{pmatrix}, \tag{3.16}$$

where u_α is the macro-velocity in cell B_α, which is given by the first term in (3.14).

Considering an arbitrary realization of the random process (3.14), we shall determine the evolution of the number of EVFs, $N_\alpha^\pm(t)$, within some fixed cell B_α. This random quantity is given by relation (3.15) defining the so-called Smoluchovsky process (Kac 1957). If $N_\alpha^\pm(t)$ EVFs of both types are placed within each cell B_α at time t, then this number can be determined at time $t+\Delta t$ as

$$N_\alpha^\pm(t+\Delta t) = \sum_{k \in J^\pm} \chi_\alpha(x_k(t+\Delta t)). \tag{3.17}$$

The coordinates of EVFs at time $t+\Delta t$ are determined from (3.14) as

$$x_k(t+\Delta t) = x_k(t) + u_{\alpha(k)}\Delta t + (2\nu)^{1/2}\eta_k(\Delta t) + o(\Delta t), \tag{3.18}$$

where $\eta_k(\Delta t)$ are the statistically independent random values of the Gaussian law with mean zero and variance $2\nu t$. The motion law (3.18) is illustrated in figure 2. Representing (3.17) as the sum of sums over each cell B_β, we obtain

$$N_\alpha^\pm(t+\Delta t) = \sum_{\beta=1}^{M} \sum_{k\in J_\beta^\pm(t)} \chi_\alpha(\boldsymbol{x}_k(t)+\boldsymbol{u}_\beta(t)\,\Delta t+(2\nu)^{1/2}\boldsymbol{\eta}_k(\Delta t)), \tag{3.19}$$

where $J_\beta^\pm(t)$ are the index sets of EVF numbers placed within the cell B_β at time t.

The collection of random numbers $\{\boldsymbol{\eta}_k(\Delta t)\}_{k\in J_\beta^\pm(t)}$ is a series of identical and independent samples. Let $Q_{\Delta S}(\boldsymbol{r})$ be the number of the vector extremities $\boldsymbol{\eta}_k(\Delta t)$ in given series that find themselves within the small area ΔS, the centre of which is denoted by the vector $\boldsymbol{r}$. This quantity can be represented in the form

$$Q_{\Delta S}(\boldsymbol{r}) = \sum_{k\in J_\beta^\pm(t)} \chi_{\Delta S}(\boldsymbol{\eta}_k(\Delta t)),$$

where $\chi_{\Delta S}(\boldsymbol{r})$ is the characteristic function of the area ΔS. Evidently, this quantity is a random value also and depends on the number of samples in the series. Note that if the number $N_\beta^\pm$ is large enough, then the quantity $Q_{\Delta S}(\boldsymbol{r})$ can be written according to the law of large numbers as

$$Q_{\Delta S}(\boldsymbol{r}) = N_\beta^\pm(t)\,a+(N_\beta^\pm(t))^{1/2}\sigma_0\,Z_\beta(N_\beta^\pm), \tag{3.20}$$

where a is the mean value of the random value $\chi_{\Delta S}(\boldsymbol{\eta}_k(\Delta t))$, σ_0 is its variance. By virtue of the Gaussian law of distribution of each value $\boldsymbol{\eta}_k(\Delta t)$ with mean zero, variance $2\nu t$ and small area ΔS ($\Delta S \ll \nu\Delta t$), then a and σ_0 are determined by

$$a \approx \frac{1}{4\pi\nu\,\Delta t}\exp(-|\boldsymbol{r}|^2/4\nu\,\Delta t)\,\Delta S, \quad a \ll 1, \tag{3.21a}$$

$$\sigma_0^2 = a(1-a) \approx a. \tag{3.21b}$$

The distribution function of the random value $Z_\beta(N_\beta^\pm)$ has approximately the Gaussian form

$$F_{N_\beta^\pm}(z) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{z} \exp(-y^2/2)\,\mathrm{d}y+O\left(\frac{1}{(N_\beta^\pm)^{1/2}}\right). \tag{3.22}$$

Let us determine the inner sum in (3.19) for given β, which allows us to find the number of EVFs passing from cell B_β into cell B_α during time Δt. This quantity is equal to $Q_{\Delta S_\alpha}(\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta-\boldsymbol{u}_\beta(t)\,\Delta t)$ with an accuracy of $O(\Delta t)$ and $O(d)$ because the velocity $\boldsymbol{u}_\beta(t)$ of all EVFs in cell B_β at time t is identical. Thus, the sum (3.19) can be written in the form

$$N_\alpha^\pm(t+\Delta t) = \sum_{\beta=1}^{M} Q_{\Delta S_\alpha}(\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta-\boldsymbol{u}_\beta(t)\,\Delta t). \tag{3.23}$$

Substituting (3.20) and (3.21) into (3.23), we obtain

$$N_\alpha^\pm(t+\Delta t) = \left(\sum_{\beta=1}^{M} \frac{N_\beta^\pm(t)}{4\pi\nu\,\Delta t}\exp\left(-\frac{|\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta-\boldsymbol{u}_\beta(t)\,\Delta t|^2}{4\nu\,\Delta t}\right)+Z_\pm(\Delta t)\right)\Delta S_\alpha, \tag{3.24}$$

where the random function $Z_\pm(\Delta t)$ is expressed by

$$Z_\pm(\Delta t) = \frac{1}{\Delta S_\alpha}\sum_{\beta=1}^{M}\left(\frac{N_\beta^\pm(t)}{4\pi\nu\,\Delta t}\exp\left[-\frac{|\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta-\boldsymbol{u}_\beta(t)\,\Delta t|^2}{4\nu\,\Delta t}\right]\Delta S_\beta\right)^{1/2} Z_\beta(N_\beta^\pm). \tag{3.25}$$

Expressing $N_\beta^\pm(t)$ through the concentrations, $N_\beta^\pm(t) = N_\pm(\boldsymbol{x}_\beta, t)\Delta S_\beta$, in (3.24) and (3.25), we obtain the next relation in the main approximation with respect to d and ϵ:

$$N_\pm(\boldsymbol{x}_\alpha, t+\Delta t) = \int \frac{\exp[-|\boldsymbol{x}_\alpha-\boldsymbol{y}-\boldsymbol{u}(\boldsymbol{y},t)\,\Delta t|^2/4\nu\,\Delta t]}{4\pi\nu\,\Delta t} N_\pm(\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y}+Z_\pm(\Delta t), \tag{3.26}$$

where $u(y,t)$ is given by (3.13). Using (3.10), we find from (3.26) that the vorticity variation in a given realization of the process during time Δt is

$$\Omega(x,t+\Delta t) = \int \frac{\exp[-|x-y-u(y,t)\,\Delta t|^2/4\nu\,\Delta t]}{4\pi\nu\,\Delta t}\Omega(y,t)\,\mathrm{d}y+\gamma_0 Z(\Delta t), \quad (3.27)$$

where $Z(\Delta t) = Z_+(\Delta t) - Z_-(\Delta t)$. The integral in (3.27) can be expanded by the Laplace method (see Dynkin 1965) into a series with respect to time step Δt as

$$\Omega(x,t)+\Delta t(-\nabla\cdot(u(x,t)\,\Omega(x,t))+\nu\,\Delta\Omega(x,t))+o(\Delta t). \quad (3.28)$$

Therefore, (3.27) can be written correct to the terms of first-order relative to the time step Δt as follows:

$$\Omega(x,t+\Delta t) = \Omega(x,t)+\Delta t(-\nabla\cdot(u(x,t)\,\Omega(x,t))+\nu\,\Delta\Omega(x,t))+\gamma_0 Z(\Delta t). \quad (3.29)$$

Relation (3.29) determines the vorticity evolution during one realization of the motion process (3.14). The vorticity in (3.29) is a random value because the trajectory x_t in space R^{2N} is a random one. This fact is taken into account in (3.29) by the random function $\gamma_0 Z(\Delta t)$, and, therefore, this equation is interpreted as stochastic one. However, the function $\gamma_0 Z(\Delta t)$ has approximately a Gaussian distribution with variance estimated by (3.22) and (3.25) as

$$\sigma^2 \approx \frac{\gamma_0^2}{\Delta S_\alpha^2}\sum_{\beta=1}^{M}\frac{N_\beta^{\pm}(t)}{4\pi\nu\,\Delta t}\exp\left[-\frac{|x_\alpha-x_\beta-u_\beta(t)\,\Delta t|^2}{4\nu\,\Delta t}\right]\Delta S_\beta \sim N_\alpha^{\pm}(t+\Delta t)\frac{\gamma_0^2}{\Delta S_\alpha^2}\sim\frac{\Gamma^2\epsilon^2}{L^4 d^2}.$$

The role of the random factor vanishes under the condition $\epsilon/d\to 0$, when $N\to\infty$, $\epsilon\to 0$. In other words, the last stochastic term in (3.29) is negligible when $\epsilon/d\to 0$, and the first regular one determines the vorticity evolution. Thus, the vorticity evolution follows the Helmholtz equation (3.1) in every realization of the random process under the above condition, and we have proved that the definitions of §§3.1 an 3.2 lead to the correct simulation of the Helmholtz equation by the continuous Markov process. It should be noted that that we have used here the limiting process $\epsilon\to 0$ and $d\to 0$ under the condition $\epsilon/d\to 0$ before the limit $\Delta t\to 0$. It is this order of limiting processes that allows us to introduce correctly the micro-level with infinite number of EVFs in a flow.

4. Two-dimensional flows of a viscous fluid in unbounded space

Here, we shall determine the general solution to the Cauchy problem for the Helmholtz equation in an unbounded viscous flow in the form of both linear and nonlinear product integrals using the continuous Markov process constructed in the previous Section.

It is known that there is no vorticity production process in unbounded two-dimensional flows of a viscous fluid, and the vorticity evolution is determined by the interactions between the convection and diffusion process. This fact is confirmed by the circulation and momentum conservation theorems expressed in the form (see Leonard 1980)

$$\Gamma = \int \Omega(x,t)\,\mathrm{d}x = \text{const}, \quad (4.1)$$

$$p = \int \Omega(x,t)\,x\,\mathrm{d}x = \text{const}. \quad (4.2)$$

In the approach developed here, the circulation conservation theorem (4.1) is interpreted as the conservation in time of the total number of each type of EVF, or there is no EVF production. The momentum conservation theorem (4.2) can be interpreted as the conservation in time of the 'centre of gravity' of the whole set of EVFs if those of γ^+ type have conditionally mass $+1$ and those of γ^- type have the same negative mass.

4.1. *Description of vorticity evolution in the form of a linear product integral*
Assume that some large enough finite number N of EVFs moves according to (3.14), and all the assumptions that were made in the derivation (3.10) hold. Then the mean value

$$\Omega(\boldsymbol{x}, t) = E_{t_0, x_0}\left[\left(\sum_{k\in J^+} - \sum_{k\in J^-}\right)\gamma_0\,\delta(\boldsymbol{x}-\boldsymbol{x}_k(t))\right] \tag{4.3}$$

is the solution for the Helmholtz equation (3.1) at $\epsilon\to 0$, $d\to 0$ ($\epsilon/d\to 0$) with initial vorticity of the form

$$\Omega_0(\boldsymbol{x}) = \left(\sum_{k\in J^+} - \sum_{k\in J^-}\right)\gamma_0\,\delta(\boldsymbol{x}-\boldsymbol{x}_{0k}), \tag{4.4}$$

where $\boldsymbol{x}_k(t)$ ($k = 1, \ldots, N$) are random trajectories of EVFs starting out from the initial point $\boldsymbol{x}_{0k}$, and E_{t_0, x_0} denotes the mean value all over the samples, the trajectories of which start from the point x_0 at time t_0, and x_0 a vector of the form (3.16). Indeed, the vorticity evolution (3.29) follows the Helmholtz equation with overwhelming probability at large numbers N in every realization of the process (in the limit $N\to\infty$ with probability equal to unity). Therefore, the mathematical expectation (4.3) will also be a solution for this equation, when $N\to\infty$.

Since the averaging functional in (4.3) depends on the coordinate of the last point of the trajectory x_t, being considered at time interval $[t_0, t]$, the mathematical expectation is determined as follows:

$$\Omega(\boldsymbol{x}, t) = \int_{R^{2N}} p(y, t, x_0, t_0)\left(\sum_{k\in J^+} - \sum_{k\in J^-}\right)\gamma_0\,\delta(\boldsymbol{x}-y_k)\,\mathrm{d}y, \tag{4.5}$$

where $p(y, t, x_0, t_0)$ is the transition probability density of the continuous Markov process (3.14). The density $p(x, t, x_0, t_0)$ obeys the forward Kolmogorov equation in the form

$$\frac{\partial p}{\partial t} + \nabla_x(pu) = \nu\,\Delta_x p, \tag{4.6}$$

where ∇_x and Δ_x are the differential operators with respect to $x\in R^{2N}$, and u is the velocity given by (3.16). Then, the function $p(x, t, x_0, t_0)$ is the Green's function for (4.6) that is expressed by the product integral (Daletsky 1962)

$$p(x, t, x_0, t_0) = \int_{(x_0, t_0)}^{(x, t)} Dx(t)\exp\left\{-\frac{1}{4\nu}\int_{t_0}^{t} |\dot{x}_\tau - u(x_\tau)|^2\,\mathrm{d}\tau\right\}, \tag{4.7}$$

where $\dot{x}_\tau = \mathrm{d}x_\tau/\mathrm{d}\tau$ and $\qquad Dx(t) = \prod_{\tau=t_0} \dfrac{\mathrm{d}x(\tau)}{\mathrm{d}\tau 4\pi\nu}.$

To evaluate the integral (4.5), introduce the transition probability density $p_k(\boldsymbol{x}, t, x_0, t_0)$ ($k = 1, \ldots, N$) of events in which the kth vortex filament arrives at point $\boldsymbol{x}$ at time t, if all EVFs are at the point x_0 at time t_0, in the form

$$p_k(\boldsymbol{x}, t, x_0, t_0) = \int_{R^{2N-2}} p(y(\boldsymbol{x})_k, t, x_0, t_0)\,\mathrm{d}_k y, \tag{4.8}$$

where we use the notation

$$\mathrm{d}_k y = \mathrm{d}y_1 \ldots \mathrm{d}y_{k-1}\,\mathrm{d}y_{k+1} \ldots \mathrm{d}y_N, \quad y(x)_k = (y_1, \ldots, y_{k-1}, x, y_{k+1}, \ldots, y_N).$$

Taking into account (4.8), expression (4.3) can be written as

$$\Omega(x,t) = \left(\sum_{k \in J^+} - \sum_{k \in J^-}\right) \gamma_0 p_k(x, t, x_0, t_0). \tag{4.9}$$

Substituting the product integral (4.7) into (4.8) and then substituting (4.8) into (4.9), we obtain

$$\Omega(x,t) = \left(\sum_{k \in J^+} - \sum_{k \in J^-}\right) \gamma_0 \int_{R^{2N-2}} \mathrm{d}_k y \int_{(x_0, t_0)}^{(y(x)_k, t)} Dx(t) \exp\left\{-\frac{1}{4\nu} \int_{t_0}^{t} |\dot{x}_\tau - u(x_\tau)|^2 \,\mathrm{d}\tau\right\}. \tag{4.10}$$

In the limiting case $\epsilon \to 0$, $d \to 0$ ($\epsilon/d \to 0$), (4.10) is the product integral determining the solution for the Helmholtz equation with initial vorticity (4.4). The main advantage of (4.10) is that it is the most simple type of product integral, i.e. linear, which is sometimes more convenient for the analytic study.

4.2. *Description of vorticity evolution in the form of a nonlinear product integral*

Let us obtain the solution of the Helmholtz equation based on (3.27) that gives the vorticity variance that occurred in a small time step Δt in one realization of stochastic process (3.14). Unlike the previous approach, we shall first take the limit of (3.27) when $\epsilon \to 0$, $d \to 0$ ($\epsilon/d \to 0$). Then, the stochastic term in (3.27) vanishes according to the estimates made above, and (3.27) takes the form

$$\Omega(x, t+\Delta t) = \int \frac{\exp[-|x-y-u(y,t)\,\Delta t|^2/4\nu\,\Delta t]}{4\pi\nu\,\Delta t} \Omega(y,t)\,\mathrm{d}y + o(\Delta t), \tag{4.11}$$

where the velocity field is determined through the Biot–Savart law

$$u(x,t) = \frac{1}{2\pi} \int \Omega(y,t) \frac{e_z \times (x-y)}{|x-y|^2} \mathrm{d}y. \tag{4.12}$$

Relation (4.11) is a nonlinear integral operator which transforms the vorticity $\Omega(x,t)$ at time t to the vorticity $\Omega(x, t+\Delta t)$ at time $t+\Delta t$. We denote this operator as $\hat{T}_{\Delta t}\,\Omega(t)$. As is proved above, the operator $\hat{T}_{\Delta t}\,\Omega(t)$ determines, in the limit as $\Delta t \to 0$, the variance of the vorticity according to the Helmholtz equation (3.1).

We shall now determine the vorticity evolution in time interval $[t_0, t]$. To this end, we take some partition q ($t_0 < t_1 < \ldots < t_m < t$) of this time interval and construct a sequence of the vorticities at the time instants t_i ($i = 1, \ldots, m$) by the recurrent method, as the product of integral operators $\hat{T}_{t_j - t_{j-1}} \Omega(t_{j-1})$ ($j = 1, \ldots, i$), of the form

$$\Omega_{i,m}(x, t_i) = \int \ldots \int p(x, t_i, x_{i-1}, t_{i-1}; u(x_{i-1}, t_{i-1})) \cdot \ldots \cdot p(x_1, t_1, x_0, t_0; u(x_0, t_0))\, \Omega(x_0, t_0)\, \mathrm{d}x_0 \ldots \mathrm{d}x_{i-1}, \tag{4.13}$$

where

$$p(x, t, y, \tau; u(y,t)) = \frac{1}{4\pi\nu(t-\tau)} \exp\left(-\frac{|x-y-u(y,t)(t-\tau)|^2}{4\nu(t-\tau)}\right), \tag{4.14}$$

and the velocity field $u(x, t_i)$ ($1 \leqslant i \leqslant m$) is determined by the vorticity $\Omega(x, t_i)$ through the Biot–Savart law (4.12).

When the intervals of the partition q are very small, i.e. $m \to \infty$ and $t_{i+1} - t_i \to 0$, expression (4.13) transforms to the nonlinear product integral

$$\Omega(x, t) = \int^{x(t)=x} Dx(t)\, \Omega(x(t_0), t_0) \overleftarrow{\exp}\left\{ -\frac{1}{4\nu} \int_{t_0}^{t} \left| \dot{x}_\tau - \frac{1}{2\pi} \int \Omega(y, t) \frac{e_z \times (x_\tau - y)}{|x_\tau - y|^2} \mathrm{d}y \right|^2 \mathrm{d}\tau \right\}. \tag{4.15}$$

Here, the notation of the nonlinear product integral is taken in accordance with that of Maslov (1976), who first introduced this version of the Wiener–Feynman integrals. The arrow over the 'exp'denotes that the evaluation of the products in (4.13) is to be made on each following step using the result of the previous one.

The nonlinear product integral (4.15) is the exact analytic solution of the Cauchy problem for the Helmholtz equation (3.1) with the initial vorticity $\Omega(x, t_0)$. This integral means the sum over all EVF trajectories passing from any point x_0 to a fixed point x during time interval $[t_0, t]$ so that the contribution of each path x_t to the sum is proportional to

$$\exp(-S[x_t]/\nu),$$

where $S[x_t]$ is a functional of the form

$$S[x_t] = \int_{t_0}^{t} |\dot{x}_\tau - u(x_\tau, \tau)|^2 \,\mathrm{d}\tau. \tag{4.16}$$

At small kinematic viscosity, the main contribution to the integral (4.15) is made by trajectories situated in the neighbourhood of extremal paths for the functional $S[x_t]$, but the contribution of other paths is negligible. The extremal path, passing through points x_0 and x, gives the minimum value of $S[x_t]$ and satisfies the Lagrange equation of the form

$$\ddot{x}_t + [\dot{x}_t \times \Omega(x_t, t)\, e_z] - \tfrac{1}{2}\nabla u^2(x_t, t) - \frac{\partial u(x_t, t)}{\partial t} = 0.$$

This property creates the basis for developing asymptotic methods for evaluation of the product integrals (see Feynman & Hibbs 1965; Maslov 1976).

When $\nu = 0$, it is necessary to take into account the contribution of the extremal paths only, for which $S[x_t]$ in (4.16) is exactly equal to zero. Each such path satisfies the equation $\dot{x}_t = u(x_t, t)$ that expresses vortex dynamics of an ideal fluid. Therefore, the solution (4.15) for unbounded viscous fluid flows transforms into this one for the vortex dynamics of an ideal fluid as $\nu \to 0$.

5. Two-dimensional flows of a viscous fluid with boundaries

Let $G(t)$ be an $n+1$-connected plane domain of a flow bounded by simple non-crossing curves $S_0(t)$, $S_1(t), \ldots, S_n(t)$, and $S_0(t)$ embraces all the other ones. Let $G_1(t), \ldots, G_n(t)$ be the inner domains embraced by the circuits $S_1(t), \ldots, S_n(t)$, and $G_0(t)$ be an external domain relative to the circuit $S_0(t)$, and, in addition, $\tilde{G}(t) = G_0(t) \cup G_1(t) \cup \ldots \cup G_n(t)$ is the external domain for a flow. We shall also take $z_s(t) = x(s, t) + \mathrm{i}y(s, t)$ as the equation for the circuits in complex parametric form. The real variable s here runs consistently round all circuits $S_0(t), S_1(t), \ldots, S_n(t)$. Denote $S(t) = S_0(t) \cup S_1(t) \cup \ldots \cup S_n(t)$. Assume also that a finite number of angular points can be placed on the circuits and introduce the boundary function $\alpha(s, t)$ which is equal to the angle between two tangents drawn to a contour at some angular point counting from the left-hand side of a circuit, and $\alpha(s, t) = \pi$ at smooth points of a circuit.

The velocity field will be then expressed in the complex form

$$w(z, \bar{z}, t) = u_y + \mathrm{i}u_x, \tag{5.1}$$

where $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$. The boundary velocity will be expressed in the form $w_s(t) = \mathrm{i}\,\mathrm{d}\bar{z}_s(t)/\mathrm{d}t$. From (2.2) and (2.3), the connection between complex velocity field and the vorticity is given by the relation

$$\frac{\partial w(z, \bar{z}, t)}{\partial \bar{z}} = \tfrac{1}{2}\Omega(z, \bar{z}, t). \tag{5.2}$$

Here, the variables (x, y) are changed to $(z, \bar{z})$ and $\partial/\partial z = \frac{1}{2}(\partial/\partial x - \mathrm{i}\,\partial/\partial y)$, $\partial/\partial \bar{z} = \frac{1}{2}(\partial/\partial x + \mathrm{i}\,\partial/\partial y)$.

The appearance of moving boundaries in a flow leads to an additional effect, in particular to the production of vorticity, besides diffusion and convection. To describe this process according to the vortex dynamics concept, it is necessary to reformulate the no-slip boundary condition (3.4) in terms of the vorticity.

5.1. *Definition of the no-slip boundary condition in terms of vorticity*

Determine all possible vorticity fields $\Omega(\boldsymbol{x}, t)$ in a flow domain at time t for which the velocity fields satisfy (3.2)–(3.4). Following the theory of Vekua (1952) (see Gakhov 1967), any continuous complex function $f(z, \bar{z}, t)$ can be expressed in the bounded domain $G(t)$ in the form

$$f(z, \bar{z}, t) = \frac{1}{2\pi \mathrm{i}} \int_{S(t)} \frac{f_s(\zeta, t)}{\zeta - z}\,\mathrm{d}\zeta + \frac{1}{\pi} \iint_{G(t)} \frac{\partial f(\zeta, \bar{\zeta}, t)}{\partial \bar{\zeta}} \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{z - \zeta}, \tag{5.3}$$

where $\zeta = \xi + \mathrm{i}\eta$, $f_s(z_s, t)$ is the boundary value of the function $f(z, \bar{z}, t)$ at the point z_s of $S(t)$, and the direction of integration along $S(t)$ in the first integral is chosen so that the domain $G(t)$ is from the left-hand side. In the limit of $z \to z_s$ in (5.3) at $z \in G(t)$ (from the side of a flow domain), the application of Sochotzky's formula for the case of angular points on an integrating contour (see Gakhov 1967) gives the following integral relation:

$$\frac{\alpha(z_s, t) f_s(z_s, t)}{2\pi} - \frac{1}{2\pi \mathrm{i}} \int_{S(t)} \frac{f_s(\zeta, t)}{\zeta - z_s}\,\mathrm{d}\zeta = \frac{1}{\pi} \iint_{G(t)} \frac{\partial f(\zeta, \bar{\zeta}, t)}{\partial \bar{\zeta}} \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{z_s - \zeta}, \tag{5.4}$$

where the contour integral is evaluated in the sense of the principal value. If in (5.3) and (5.4) the velocity (5.1) replaces the function $f(z, \bar{z}, t)$, and the vorticity replaces $\partial w(z, \bar{z}, t)/\partial \bar{z}$ according to (5.2), then (5.4) transforms into the integral equation relative to the vorticity $\Omega(z, \bar{z}, t)$ at the given boundary velocity $w_s(z_s, t)$

$$\frac{\alpha(z_s, t)\, w_s(z_s, t)}{2\pi} - \frac{1}{2\pi \mathrm{i}} \int_{S(t)} \frac{w_s(\zeta, t)}{\zeta - z_s}\,\mathrm{d}\zeta = \frac{1}{2\pi} \iint_{G(t)} \frac{\Omega(\xi, \eta, t)}{z_s - \zeta}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{5.5}$$

and (5.3) gives the flow velocity $w(z, \bar{z}, t)$ at the prescribed vorticity $\Omega(z, \bar{z}, t)$ in the form

$$w(z, \bar{z}, t) = \frac{1}{2\pi \mathrm{i}} \int_{S(t)} \frac{w_s(\zeta, t)}{\zeta - z}\,\mathrm{d}\zeta + \frac{1}{2\pi} \iint_{G(t)} \frac{\Omega(\xi, \eta, t)}{z - \zeta}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{5.6}$$

Expression (5.6) generalizes the Biot–Savart law to the case of flows with boundaries. Equation (5.5) and expression (5.6) solve the problem of reformulating the no-slip boundary condition in terms of the vorticity. Note that (5.5) and (5.6) can be generalized to the case when the external boundary $S_0(t)$ is absent and the flow is running from infinity with the velocity $w_\infty(t)$. In these flows, the boundary $S_0(t)$ is to be removed to infinity, and the boundary condition is to be written on it in the form $w_s|_{S_0} = w_\infty(t)$. As a result, we obtain

$$\frac{\alpha(z_s, t)\, w_s(z_s, t)}{2\pi} - \frac{1}{2\pi i}\int_{S(t)} \frac{w_s(\zeta, t)}{\zeta - z_s}\,\mathrm{d}\zeta = \frac{1}{2\pi}\iint_{G(t)} \frac{\Omega(\xi, \eta, t)}{z_s - \zeta}\,\mathrm{d}\xi\,\mathrm{d}\eta + w_\infty(t), \tag{5.7}$$

$$w(z, \bar{z}, t) = \frac{1}{2\pi \mathrm{i}}\int_{S(t)} \frac{w_s(\zeta, t)}{\zeta - z}\,\mathrm{d}\zeta + \frac{1}{2\pi}\iint_{G(t)} \frac{\Omega(\xi, \eta, t)}{z - \zeta}\,\mathrm{d}\xi\,\mathrm{d}\eta + w_\infty(t). \tag{5.8}$$

Thus, (5.5) and (5.6) will be used further without loss of generality.

5.2. *Vorticity evolution in flows with boundaries*

The physical pattern of vorticity convection and diffusion based on the stochastic motion of EVFs in the continuous Markov process is the same as for unbounded flows. However, in this case, it is also necessary to determine the behaviour of all EVFs on the boundary $S(t)$ and to introduce the production of two types of EVFs to satisfy (5.5). The most simple method of satisfying this boundary condition is the construction of the generalized Markov process (see Portenko 1982). This process is first characterized by its continuation into the outer domain $\tilde{G}(t)$. In other words, it is necessary to allow EVFs to run through boundaries outside a flow domain and to move randomly in this domain. This assumption is, of course, not correct from physical point of view because flow is absent within outer domains. Since this method does not change the Helmholtz equation in the flow domain $G(t)$ and the solution of the problem can be written in the more simple analytic form under the given boundary conditions, we choose this approach.

To continue the Markov process into the domains $G_0(t), G_1(t), \ldots, G_n(t)$, it is necessary to continue the velocity field $w(z, \bar{z}, t)$ inside them. In this case, the requirement of the solenoidality of a continuous field can be failed. The most simple method of continuation is the solution of Dirichlet's problem $\Delta w = 0$ in domains $G_0(t), G_1(t), \ldots, G_n(t)$ with boundary condition $w_s(z_s, t)$. Note that (5.8) can be used to calculate the velocity field in the whole space, when $w_s(z_s, t) = 0$.

To construct the generalized Markov process, the functions $\boldsymbol{u}_\epsilon(\boldsymbol{x}, t)$, $c_\epsilon(\boldsymbol{x}, t)$, $q_\epsilon(\boldsymbol{x}, t)$ are to be given in the ϵ-vicinity of boundaries (see (2.6)) describing the reflection of EVFs because of their strong motion with the velocity $\boldsymbol{u}_\epsilon(\boldsymbol{x}, t)$ near surfaces, their absorption with the probability density $c_\epsilon(\boldsymbol{x}, t)$ and their production on boundaries with the intensity $q_\epsilon(\boldsymbol{x}, t)$. In this connection, the function $q(s\, t)$,

$$\lim_{\epsilon \to 0} q_\epsilon(\boldsymbol{x}, t) = q(s, t)\, \delta_{S(t)}(\boldsymbol{x}),$$

can be either positive or negative. If $q(s, t) > 0$ at some boundary point, then EVFs of intensities γ^+ are produced, but if $q(s, t) < 0$, the intensity of vortices produced will be γ^-.

To satisfy (5.5), only the process of vorticity production on boundaries is of importance, and the choice of $\boldsymbol{u}_\epsilon(\boldsymbol{x}, t)$ and $c_\epsilon(\boldsymbol{x}, t)$ (reflection and absorption) can be, to some extent, arbitrary. However, the vortex production intensity depends on a reflection and an absorption. From an analytical point of view, the most simple generalized Markov process will be such that there will be no reflection and an absorption, and EVFs could cross boundaries freely. We consider just this case.

Let the concentrations of both types of EVFs be given in the whole space at time t so that the vorticity $\Omega(\boldsymbol{x}, t)$, determined by (3.10), satisfies (5.5) at the boundary velocity $\boldsymbol{v}_s(t)$. In addition, the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ is found using (5.6) within the flow domain $G(t)$ and is calculated by appropriate continuation of the velocity field (5.6) in the external domain $\tilde{G}(t)$. In a small time interval $[t, t+\Delta t]$ each EVF moves with local velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ independently of the others according to (3.14). Therefore, the vorticity $\Omega'(\boldsymbol{x}, t+\Delta t)$, created by these EVFs in the whole space at time $t+\Delta t$, is calculated as

$$\Omega'(\boldsymbol{x}, t+\Delta t) = \int_{R^2} p(\boldsymbol{x}, t+\Delta t, \boldsymbol{y}, t; \boldsymbol{u}(\boldsymbol{y}, t))\, \Omega(\boldsymbol{y}, t)\, \mathrm{d}\boldsymbol{y}, \tag{5.9}$$

where $p(\boldsymbol{x}, t+\Delta t, \boldsymbol{y}, t; \boldsymbol{u}(\boldsymbol{y}, t))$ is given by (4.14). The new vorticity $\Omega'(\boldsymbol{x}, t+\Delta t)$ does not satisfy (5.5) at time $t+\Delta t$. Then a number of EVFs of both types with total intensity $q(s, t)\,\Delta t$ will be produced on boundaries at the time interval $[t, t+\Delta t]$. These filaments will diffuse and create some vorticity at time $t+\Delta t$ of the form

$$\Omega''(\boldsymbol{x}, t+\Delta t) = \oint\!\!\oint_{S(t)} p(\boldsymbol{x}, t+\Delta t, \boldsymbol{y}_s, t; \boldsymbol{u}(\boldsymbol{y}_s, t))\, q(s, t)\, \Delta t\, \mathrm{d}S_y. \tag{5.10}$$

Then the total vorticity in a flow at the time $t+\Delta t$ is the sum (5.9) and (5.10):

$$\Omega(\boldsymbol{x}, t+\Delta t) = \int_{R^2} p(\boldsymbol{x}, t+\Delta t, \boldsymbol{y}, t; \boldsymbol{u}(\boldsymbol{y}, t))\, \Omega(\boldsymbol{y}, t)\, \mathrm{d}\boldsymbol{y} + \oint\!\!\oint_{S(t)} p(\boldsymbol{x}, t+\Delta t, \boldsymbol{y}_s, t; \boldsymbol{u}(\boldsymbol{y}_s, t))\, q(s, t)\, \Delta t\, \mathrm{d}S_y. \tag{5.11}$$

The intensity $q(s, t)$ is to be chosen so that the vorticity $\Omega(\boldsymbol{x}, t+\Delta t)$ (5.11) would satisfy (5.5) with boundary velocity $\boldsymbol{v}_s(t+\Delta t)$ at time $t+\Delta t$. Therefore, $q(s, t)$ depends on $\Omega(\boldsymbol{x}, t)$, $\boldsymbol{v}_s(t)$, and $\boldsymbol{v}_s(t+\Delta t)$. Note that the dependence of $q(s, t)$ on the boundary velocity $\boldsymbol{v}_s(t+\Delta t)$ at time $t+\Delta t$ expresses the influence of the boundary acceleration at time t on this quantity. Expression (5.11) gives an iterative method for the transition from $\Omega(\boldsymbol{x}, t)$ to $\Omega(\boldsymbol{x}, t+\Delta t)$, when this dependence is established.

Using expansion (3.28) of the integral (5.9) and an expression of the form

$$\lim_{\Delta t \to 0} \oint\!\!\oint_S \frac{\exp[-|\boldsymbol{x}-\boldsymbol{y}_s|^2/4\nu\Delta t]}{4\pi\nu\,\Delta t} q(s, t)\, \mathrm{d}S_y = q(s, t)\, \delta_s(\boldsymbol{x}), \tag{5.12}$$

we find that the vorticity evolution satisfies, in the limit $\Delta t \to 0$, the equation

$$\frac{\partial \Omega}{\partial t} + \nabla \cdot (\boldsymbol{u}\Omega) = \nu\, \Delta\Omega + q(s, t)\, \delta_S(\boldsymbol{x}). \tag{5.13}$$

In the following time intervals $[t+k\,\Delta t, t+(k+1)\,\Delta t]$ $(k = 1, 2, \ldots)$, this process must be repeated. Successive application of the nonlinear integral operator (5.11) to the initial vorticity allows, by analogy with (4.13), the vorticity $\Omega(\boldsymbol{x}, t_0+k\,\Delta t)$ to be expressed through $\Omega(\boldsymbol{x}, t_0)$ and the nonlinear product integral at $\Delta t \to 0$, $k \to \infty$ $(k\,\Delta t = T = \text{const})$ to be obtained in the form

$$\Omega(\boldsymbol{x}, t) = \int^{\boldsymbol{x}(t)=\boldsymbol{x}} D\boldsymbol{x}(t)\, \Omega(\boldsymbol{x}(t_0), t_0)\, \overleftarrow{\exp}\left\{-\frac{1}{4\nu}\int_{t_0}^{t} |\dot{\boldsymbol{x}}_{\tau'} - \boldsymbol{u}(\boldsymbol{x}_\tau, t)|^2\, \mathrm{d}\tau\right\} + \int_{t_0}^{t} \mathrm{d}\tau \int^{\boldsymbol{x}(t)=\boldsymbol{x}} D\boldsymbol{x}(t)\, q(s, \tau)\, \delta_S(\boldsymbol{x}_\tau)\, \overleftarrow{\exp}\left\{-\frac{1}{4\nu}\int_{t_0}^{t} |\dot{\boldsymbol{x}}_\tau - \boldsymbol{u}(\boldsymbol{x}_\tau, \tau)|^2\, \mathrm{d}\tau\right\}, \tag{5.14}$$

where $\boldsymbol{u}(\boldsymbol{x}, t)$ is calculated through $\Omega(\boldsymbol{x}, t)$ by (5.6) in the flow domain $G(t)$ and by non-stop continuation into $\tilde{G}(t)$ at every time instant.

The nonlinear product integral (5.14) is the solution of the Cauchy problem for equation (5.13). This equation transforms into the Helmholtz equation (3.1) in the flow domain $G(t)$, and the conditions (3.2)–(3.4) are satisfied by the definitions of $\boldsymbol{u}(\boldsymbol{x}, t)$ and $q(s, t)$. Thus, it is proved that PIM can be used to simulate the vortex dynamics of a viscous fluid with moving boundaries in a flow. In our calculations, not reproduced here, the convergence of product integrals (4.15) and (5.14) is proved under wide assumptions.

5.3. *An equation for the intensity of boundary vortex production*

We now determine the dependence of the vortex production intensity $q(s, t)$ on the vorticity in a flow, together with the boundary velocity and boundary acceleration. Assuming that the vorticity $\Omega(\boldsymbol{x}, t)$ satisfies the condition (5.5) with boundary velocity $w_s(t)$ at time t, we write (5.5) at time $t+\Delta t$ with known boundary velocity $w_s(t+\Delta t)$, substituting $\Omega(\boldsymbol{x}, t+\Delta t)$ accordingly (5.11). Linearizing the expression obtained with respect to Δt, we find an equation for the unknown intensity $q(s, t)$ in the form

$$A(z_s, t+\Delta t) - \frac{\alpha(z_s, t+\Delta t)}{2\pi}\tilde{q}_{\Delta t}(z_s, t) + \frac{1}{2\pi \mathrm{i}}\int_{S(t+\Delta t)} \frac{\tilde{q}_{\Delta t}(\zeta, t)}{\zeta - z_s}\,\mathrm{d}\zeta = 0, \qquad (5.15)$$

where the following notation is introduced:

$$A(z_s, t+\Delta t) = -\frac{\alpha(z_s, t+\Delta t)}{2\pi} w_s(z_s, t+\Delta t) + \frac{1}{2\pi \mathrm{i}}\int_{S(t+\Delta t)} \frac{w_s(\zeta, t+\Delta t)}{\zeta - z_s}\,\mathrm{d}\zeta$$
$$+ \frac{1}{2\pi}\iint_{G(t+\Delta t)} \frac{\hat{T}_{\Delta t}\,\Omega(t, \xi, \eta)}{z_s - \zeta}\,\mathrm{d}\xi\,\mathrm{d}\eta, \qquad (5.16)$$

$$\tilde{q}_{\Delta t}(z_s, t) = \frac{q_{\Delta t}(z_s, t)}{2n(z_s, t+\Delta t)}, \quad q_{\Delta t}(z_s, t) = q(s, t)\,\Delta t. \qquad (5.17)$$

Here, $n(z_s, t+\Delta t) = n_x + \mathrm{i} n_y$ is the inward complex normal to the boundary $S(t+\Delta t)$ relatively to the flow domain $G(t+\Delta t)$, and $\hat{T}_{\Delta t}\,\Omega(t, \xi, \eta)$ denotes the right-hand side of (5.9).

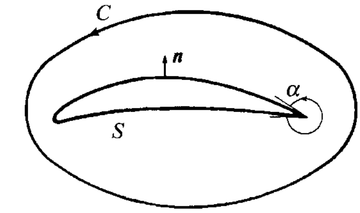
The term $A(z_s, t+\Delta t)$ in equation (5.15), determined by (5.16), represents the velocity imbalance induced by the vorticity (5.9) on the boundary $S(t+\Delta t)$ at time $t+\Delta t$. The other terms in (5.15) determine the velocity induced on the boundary $S(t+\Delta t)$ by the EVFs produced during the time interval $[t, t+\Delta t]$ that compensates the imbalance $A(z_s, t+\Delta t)$ that arises.

The solution of (5.15) is to be determined under the condition

$$\mathrm{Im}\,[\tilde{q}_{\Delta t}(z_s, t)\, n(z_s, t+\Delta t)] = 0, \qquad (5.18)$$

which means that the intensity $q(s, t)$, expressed by (5.17) through $\tilde{q}_{\Delta t}(z_s, t)$, is the real function of boundary points. Also, it is necessary to demand that the function $q(s, t)$ is single-valued.

However, these last two demands do not ensure the uniqueness of a solution of the integral equation (5.15). It will be shown later that the solution of (5.15) depends on n real arbitrary constants under the conditions (5.18). For any choice of these constants, the vortex production intensity $q(s, t)$ requires that the expression (5.11) satisfies the Helmholtz equation (3.1) and the no-slip conditions in the form (5.5). In other words, the solution of the Helmholtz equation in a plane flow is not unique under the no-slip conditions on moving boundaries.



$$\oint_C \frac{\nabla p}{\rho_0}\, \mathrm{d}x = 0$$

FIGURE 3. The condition of the pressure univalent representation is described as the zero value of the contour integral from the pressure gradient around an arbitrary contour C; S is the boundary of a body in the flow, $\boldsymbol{n}$ is the inward unit normal vector relative to the flow and α is the angle between two tangents at an angular point of the boundary.

This connected with a mathematical problem arising when the transition is made from Navier–Stokes equations to the Helmholtz equation. To calculate the pressure from the Navier–Stokes equations after determining the velocity from the Helmholtz equation, it is necessary to integrate the single-valued pressure gradient. However, in a multi-connected flow domain such integration can lead to a multi-valued pressure. This many-valued pressure vanishes at the transition from the Navier–Stokes equations to the Helmholtz equation initiating, however, the non-uniqueness of solutions of the latter. Therefore, from the mathematical point of view the Navier–Stokes and the Helmholtz equations do not have a unique solution without some additional restrictions on the pressure. From the physical standpoint the pressure is a single-valued function, hence some additional conditions must be put on the solution for the Helmholtz equation. These conditions are to be formulated on the intensity of boundary vortex production, which allows (5.15) to be solved by the only way.

The condition of a single-valued pressure can be written in a multi-connected flow domain in accordance with the Navier–Stokes equations in an incompressible fluid in the form of n contour integrals as

$$\oint_{C_k} \left(\frac{\partial \boldsymbol{u}}{\partial t} + \Omega(\boldsymbol{e}_z \times \boldsymbol{u}) - \nu \Delta \boldsymbol{u} \right) \mathrm{d}\boldsymbol{x} = 0, \quad k = 1, \dots, n, \tag{5.19}$$

where C_k is an arbitrary contour in the flow domain which embraces the boundary $S_k(t)$ (see figure 3). Using the Stokes formula and the representation of the simple layer, $\delta_s(\boldsymbol{x})$, in the form (5.12), the contour integrals (5.19) can be reduced due to (5.13) to integrating along the boundaries $S_k(t)$, $k = 1, \dots, n$:

$$\oint_{S_k(t)} \dot{\boldsymbol{v}}_s \, \mathrm{d}\boldsymbol{x} = \oint_{S_k(t)} \left(\left(1 - \frac{\alpha(s,t)}{2\pi} \right) q(s,t) - \nu \frac{\partial \Omega}{\partial n} \right) \mathrm{d}s. \tag{5.20}$$

Using the initial assumptions about the continuation of the stochastic process beyond a flow domain and the definitions of $q_{\Delta t}(z_s, t)$ and $\hat{T}_{\Delta t}\Omega(t)$, these conditions can be written in the form of differentials at small time step Δt as

$$\left(\oint_{S_k(t+\Delta t)} \boldsymbol{v}_s(t+\Delta t)\, \mathrm{d}\boldsymbol{x} + \iint_{G_k(t+\Delta t)} \hat{T}_{\Delta t}\Omega(t)\, \mathrm{d}x \right) - \left(\oint_{S_k(t)} \boldsymbol{v}_s(t)\, \mathrm{d}\boldsymbol{x} + \iint_{G_k(t)} \Omega(t)\, \mathrm{d}x \right)$$
$$= \oint_{S_k(t)} \left(1 - \frac{\alpha(s,t)}{2\pi} \right) q_{\Delta t}(z_s, t)\, \mathrm{d}s, \quad k = 1, \dots, n. \tag{5.21}$$

These conditions mean that the total intensity of EVFs produced on every circuit $S_k(t)$ and arriving in a flow domain in time interval $[t, t+\Delta t]$, as expressed by the right-hand side of (5.21), is equal to the sum of the variation of the total EVF intensity in the domain $G_k(t)$ and the variation of the circulation

$$\oint_{S_k(t)} \boldsymbol{v}_s(t)\,\mathrm{d}\boldsymbol{x}$$

in the same time interval.

Thus, the solution of (5.15) is to be found under conditions (5.18) and (5.21), with the requirement that the function $\tilde{q}_{\Delta t}(z_s, t)$ is single-valued.

5.4. *Determination of the intensity of boundary vortex production*

Let us introduce the new unknown function of boundary points

$$\phi(z_s) = \tilde{q}_{\Delta t}(z_s, t) + w_s(z_s, t+\Delta t). \tag{5.22}$$

Then (5.15) takes the form

$$-\frac{\alpha(z_s, t+\Delta t)}{2\pi}\phi(z_s) + \frac{1}{2\pi \mathrm{i}}\int_{S(t+\Delta t)} \frac{\phi(\zeta)}{\zeta - z_s}\mathrm{d}\zeta = -B(z_s, t+\Delta t), \tag{5.23}$$

where $B(z_s, t+\Delta t)$ is the boundary value of the analytic function in the domain $\tilde{G}(t+\Delta t)$ given as

$$B(z, t+\Delta t) = \frac{1}{2\pi}\iint_{G(t+\Delta t)} \frac{\hat{T}_{\Delta t}\,\Omega(t, \xi, \eta)}{z-\zeta}\mathrm{d}\xi\,\mathrm{d}\eta. \tag{5.24a}$$

If the external boundary S_0 is absent, then the function $B(z, t+\Delta t)$ has the form

$$B(z, t+\Delta t) = \frac{1}{2\pi}\iint_{G(t+\Delta t)} \frac{\hat{T}_{\Delta t}\,\Omega(t, \xi, \eta)}{z-\zeta}\mathrm{d}\xi\,\mathrm{d}\eta + w_\infty(t+\Delta t). \tag{5.24b}$$

The additional term $w_\infty(t+\Delta t)$ arises due to the evaluation of the contour integral in (5.16) along the infinite circuit S_0.

From the analytic functions theory, Sochotzky's formula and (5.23) it follows that in the domain $\tilde{G}(t+\Delta t)$ we have

$$B(z, t+\Delta t) = -\frac{1}{2\pi \mathrm{i}}\int_{S(t+\Delta t)} \frac{\phi(\zeta)}{\zeta - z}\mathrm{d}\zeta. \tag{5.25}$$

Thus, the problem of a solution for (5.23) reduces to the well-known one in which a given analytic function $B(z, t+\Delta t)$ is expressed by the Cauchy's integral in $\tilde{G}(t+\Delta t)$ (see Gakhov 1967).

The solution of this problem can be presented in the form

$$\phi(z_s) = B(z_s, t+\Delta t) + Q(z_s, t+\Delta t), \tag{5.26}$$

where $Q(z_s, t+\Delta t)$ is the boundary value of a single-valued analytic function which is given by the same integral (5.25), but in a multi-connected flow domain $G(t+\Delta t)$. Expressing $\tilde{q}_{\Delta t}(z_s, t)$ from (5.22), where $\phi(z_s)$ is determined by (5.26), and substituting $\tilde{q}_{\Delta t}(z_s, t)$ into (5.18), we obtain the Hilbert boundary condition for $Q(z, t+\Delta t)$ on the boundary $S(t+\Delta t)$:

$$\mathrm{Im}\,[Q(z, t+\Delta t)\,n(z_s, t+\Delta t)] = \mathrm{Im}\,[(-B(z_s, t+\Delta t) + w_s(z_s, t+\Delta t))\,n(z_s, t+\Delta t)]. \tag{5.27}$$

If a solution of the Hilbert problem (5.27) is known, then the solution of (5.15) can be written as

$$q_{\Delta t}(z_s, t) = 2n(z_s, t+\Delta t)(Q(z_s, t+\Delta t) + B(z_s, t+\Delta t) - w_s(z_s, t+\Delta t)). \quad (5.28)$$

The term $q_{\Delta t}(z_s, t)$, constructed in (5.28), is a single-valued function of boundary points satisfying (5.18), but it has to satisfy the conditions (5.21) also. Thus, the problem of the determination of the intensity $q_{\Delta t}(z_s, t)$ reduces to the solution of the Hilbert problem (5.27) in the multi-connected domain $G(t+\Delta t)$ under n conditions (5.21).

The solution of the inhomogeneous Hilbert problem (5.27), according to general theory (see Gakhov 1967), depends on n arbitrary real constants as the index of the complex normal $n(z_s, t+\Delta t)$ on the boundary $S(t+\Delta t)$ is equal to $1-n$. These constants are to be determined from n conditions (5.21), and the intensity (5.28) is, thereby, calculated by the only way. The Hilbert problem (5.27) has a hydrodynamic interpretation in the framework of the dynamics of an ideal fluid that permits an analytic form of the solution to be written in the most simple way and, on the other hand, gives the physical interpretation of the solution (5.28) for the intensity $q_{\Delta t}(z_s, t)$.

In this interpretation, (5.27) is the no-flow condition on the boundary $S(t+\Delta t)$, which moves with velocity $w_s(z_s, t+\Delta t)$ in an ideal flow having vorticity $\hat{T}_{\Delta t}\Omega(t)$ in $G(t+\Delta t)$. So, $B(z, t+\Delta t)$, determined by (5.24) in $G(t+\Delta t)$, is interpreted as a complex velocity in the form (5.1), created by the vorticity $\hat{T}_{\Delta t}\Omega(t)$ accordingly to the Biot–Savart law, and $Q(z, t+\Delta t)$ is the complex velocity of potential incompressible flow in $G(t+\Delta t)$, which is to be added to the velocity $B(z, t+\Delta t)$ to satisfy the no-flow boundary condition (5.27).

The potential of the velocity $Q(z, t+\Delta t)$ is an analytic function in the flow domain $G(t+\Delta t)$. However, it can be a multi-valued function in the case of a multi-connected domain that corresponds with circulation flows around the circuits $S_k(t+\Delta t)$ $(k = 1, \dots, n)$ in an ideal fluid with the velocity $B(z, t+\Delta t) + Q(z, t+\Delta t)$. Therefore, the unknown single-valued velocity $Q(z, t+\Delta t)$ can be written in the form

$$Q(z, t+\Delta t) = 2\mathrm{i}\frac{\partial \Phi(z)}{\partial z} + \sum_{k=1}^{n} \frac{\Gamma_k}{2\pi(z-z_k)}, \quad (5.29)$$

where $\Phi(z)$ is an unknown real harmonic function in $G(t+\Delta t)$, z_k are some arbitrary fixed points in the domains $G_k(t+\Delta t)$ and Γ_k are some circulations around the circuits $S_k(t+\Delta t)$. Substituting (5.29) into (5.27), we obtain the boundary condition for the potential $\Phi(z)$ as

$$\left.\frac{\partial \Phi(z)}{\partial n}\right|_{S(t+\Delta t)} = \operatorname{Im}\left[\left(-B(z_s, t+\Delta t) + w_s(z_s, t+\Delta t) - \sum_{k=1}^{n} \frac{\Gamma_k}{2\pi(z_s - z_k)}\right) n(z_s, t+\Delta t)\right]. \quad (5.30)$$

Thus, the determination of the potential $\Phi(z)$ consists in solving the Neuman problem for Laplace's equation in the domain $G(t+\Delta t)$. This problem is known (Courant & Hilbert 1953) to have a unique solution in any domain that can be expressed through the Green's function $H(z, \zeta)$ in the form

$$\Phi(z) = \oint_{S(t-\Delta t)} H(z, \zeta)\left\{\operatorname{Im}\left[\left(-B(\zeta, t+\Delta t) + w_s(\zeta, t+\Delta t) - \sum_{k=1}^{n} \frac{\Gamma_k}{2\pi(\zeta - z_k)}\right) n(\zeta, t+\Delta t)\right]\right\} \mathrm{d}S_\zeta.$$

Note that the solvability condition for the problem (5.30), in the case of finite flow domain, is the natural condition that total flow volume is maintained in time. Substituting this expression into (5.29), we obtain the solution for the Hilbert problem (5.27) as follows:

$$Q(z,t+\Delta t) = \oint_{S(t+\Delta t)} 2\mathrm{i}\frac{\partial H(z,\zeta)}{\partial z}\left\{\mathrm{Im}\left[\left(-B(\zeta,t+\Delta t)+w_s(\zeta,t+\Delta t)\right.\right.\right.$$
$$\left.\left.\left.-\sum_{k=1}^{n}\frac{\Gamma_k}{2\pi(\zeta-z_k)}\right)n(\zeta,t+\Delta t)\right]\right\}\mathrm{d}s_\zeta+\sum_{k=1}^{n}\frac{\Gamma_k}{2\pi(z-z_k)}. \quad (5.31)$$

This solution depends on n arbitrary constants Γ_k which are to be determined by the substitution (5.28), where $Q(z,t+\Delta t)$ is given by (5.31), into n contour integrals (5.21). Direct calculations, using the analytical properties of the functions $B(z,t+\Delta t)$ and $H(z,\zeta)$, give the following expression for these constants:

$$\Gamma_k = \iint_{G_k(t+\Delta t)} T_{\Delta t}\,\Omega(t)\,\mathrm{d}x - \iint_{G_k(t)} \Omega(t)\,\mathrm{d}x - \oint_{S_k(t)} v_s(t)\,\mathrm{d}x. \quad (5.32)$$

Substituting (5.32) into (5.31) and then into (5.28), we obtain the solution of (5.15) satisfying (5.18) and (5.21) as

$$q_{\Delta t}(z_s,t) = 2n(z_s,t+\Delta t)\left[B(z_s,t+\Delta t)-w_s(z_s,t+\Delta t)+\oint_{S(t+\Delta t)} 2\mathrm{i}\frac{\partial H(z_s,\zeta)}{\partial z}\right.$$
$$\times\{\mathrm{Im}\,[(-B(\zeta,t+\Delta t)+w_s(\zeta,t+\Delta t))\,n(\zeta,t+\Delta t)]\}\,\mathrm{d}s_\zeta$$
$$+\frac{1}{2\pi}\sum_{k=1}^{n}\left(\iint_{G_k(t+\Delta t)} \hat{T}_{\Delta t}\,\Omega(t)\,\mathrm{d}x-\iint_{G_k(t)}\Omega(t)\,\mathrm{d}x-\oint_{S_k(t)} v_s(t)\,\mathrm{d}x\right)$$
$$\left.\times\left(-\oint_{S(t+\Delta t)} 2\mathrm{i}\frac{\partial H(z_s,\zeta)}{\partial z}\mathrm{Im}\left[\frac{n(\zeta,t+\Delta t)}{\zeta-z_k}\right]\mathrm{d}s_\zeta+\frac{1}{z_s-z_k}\right)\right], \quad (5.33)$$

where $B(z_s,t+\Delta t)$ is given by (5.24). Note that the intensity $q_{\Delta t}(z_s,t)$ in (5.33) can have integrable singularities at angular points of circuits.

Taking into account the hydrodynamic interpretation of $B(z,t+\Delta t)$ and $Q(z,t+\Delta t)$, we obtain that the intensity (5.33) of EVFs produced on the boundary $S(t)$ in the time interval $[t,t+\Delta t]$ is equal to double the relative tangent velocity that appears on the boundary $S(t+\Delta t)$ at time $t+\Delta t$ in an ideal flow, having vorticity $\hat{T}_{\Delta t}\,\Omega(t)$ and circulations (5.32) around circuits $S_k(t+\Delta t)$ and satisfying no-flow boundary conditions on these circuits moving with boundary velocity $w_s(z_s,t+\Delta t)$. If the external boundary S_0 is absent, then this ideal flow has the velocity $w_\infty(t+\Delta t)$ at infinity. This hydrodynamic sense of the intensity $q_{\Delta t}(z_s,t)$ is helpful to use in numerical simulations of vorticity evolution. Note that the mathematical expression for the intensity $q_{\Delta t}(z_s,t)$ and its hydrodynamic sense consequently have to be changed on the introduction of the absorbing and reflecting EVFs at boundaries.

As a result, substituting (5.33) into the product integral (5.14), we find that the vortex dynamics of a viscous fluid with moving boundaries within a flow is unambiguously described by the nonlinear product integral.

To calculate the intensity $q_{\Delta t}(z_s,t)$ from (5.33), it is necessary to know an analytic expression of the Green's function $H(z,\zeta)$. This can be illustrated by two examples. The first one is a fluid flow past a body. Let $\theta(z,t)$ be the conformal map of the flow domain $G(t)$ into the interior of the circle $|\theta| \geqslant 1$. Then the solution of (5.33) can be written in the form

$$q_{\Delta t}(z_s, t) = 2n(z_s, t+\Delta t)\left[\frac{\theta'(z_s, t+\Delta t)}{2\pi}\iint_{G(t+\Delta t)} \frac{\hat{T}_{\Delta t}\,\Omega(t,\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{\theta(z_s, t+\Delta t)-\theta(\zeta, t+\Delta t)}\right.$$
$$-\frac{\theta'(z_s, t+\Delta t)}{2\pi}\iint_{G(t+\Delta t)} \frac{\hat{T}_{\Delta t}\,\Omega(t,\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{\theta(z_s, t+\Delta t)-1/[\theta(\zeta, 1+\Delta t)]}$$
$$+\frac{\theta'(z_s, t+\Delta t)}{2\pi\theta(z_s, t+\Delta t)}\left(\iint_{G(t)}\Omega(t)\,\mathrm{d}x-\oint_{S(t)} v_s(t)\,\mathrm{d}x\right)$$
$$\left.-\frac{\theta'(z_s, t+\Delta t)}{2\pi \mathrm{i}\theta(z_s, t+\Delta t)}\oint_{S(t+\Delta t)}\frac{\theta(\zeta, t+\Delta t)+\theta(z_s, t+\Delta t)}{\theta(\zeta, t+\Delta t)-\theta(z_s, t+\Delta t)}\mathrm{Im}\,[w_s(\zeta, t+\Delta t)\,n(\zeta, t+\Delta t)]\,\mathrm{d}s_\zeta\right]$$
$$-2\,\mathrm{Re}\,[w_s(z_s, t+\Delta t)\,n(z_s, t+\Delta t)].$$

The second case is the flow within the singly connected domain $G(t)$, and $\theta(z, t)$ here denotes the conformal map of the flow domain within the circle $|\theta| \leqslant 1$. The solution of (5.33) can be written as

$$q_{\Delta t}(z_s, t) = 2n(z_s, t+\Delta t)\left[\frac{\theta'(z_s, t+\Delta t)}{2\pi}\iint_{G(t+\Delta t)} \frac{\hat{T}_{\Delta t}\,\Omega(t,\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{\theta(z_s, t+\Delta t)-\theta(\zeta, t+\Delta t)}\right.$$
$$-\frac{\theta'(z_s, t+\Delta t)}{2\pi}\iint_{G(t+\Delta t)} \frac{\hat{T}_{\Delta t}\,\Omega(t,\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{\theta(z_s, t+\Delta t)-1/[\theta(\xi, t+\Delta t)]}$$
$$\left.-\frac{\theta'(z_s, t+\Delta t)}{2\pi \mathrm{i}\theta(z_s, t+\Delta t)}\oint_{S(t+\Delta t)}\frac{\theta(\zeta, t+\Delta t)+\theta(z_s, t+\Delta t)}{\theta(\zeta, t+\Delta t)-\theta(z_s, t+\Delta t)}\mathrm{Im}\,[w_s(\zeta, t+\Delta t)\,n(\zeta, t+\Delta t)]\,\mathrm{d}s_\zeta\right].$$
$$-2\,\mathrm{Re}\,[w_s(z_s, t+\Delta t)\,n(z_s, t+\Delta t)].$$

5.5. *The vortex dynamics theorem for a flow with moving boundaries*

Let us derive the vortex dynamics theorem for viscous fluid flows with moving boundaries. Using (5.33) one can prove by direct calculations the following relation:

$$\oint_{S(t)} \dot{v}_s\,\mathrm{d}x = \oint_{S(t)}\left(\left(1-\frac{\alpha(s,t)}{2\pi}\right)q(s,t)-\nu\frac{\partial\Omega}{\partial n}\right)\mathrm{d}s. \tag{5.34}$$

As the intensity $q_{\Delta t}(z_s, t)$ in (5.33) satisfies (5.20) on each of inner boundaries, from (5.34) it follows that (5.20) is satisfied on the external boundary $S_0(t)$, if it is in a flow, in the same form:

$$\oint_{S_0(t)} \dot{v}_s\,\mathrm{d}x = \oint_{S_0(t)}\left(\left(1-\frac{\alpha(s,t)}{2\pi}\right)q(s,t)-\nu\frac{\partial\Omega}{\partial n}\right)\mathrm{d}s. \tag{5.35}$$

By analogy to (5.21) this condition can be rewritten as

$$\left(\oint_{S_0(t+\Delta t)} v_s(t+\Delta t)\,\mathrm{d}x+\iint_{G_0(t+\Delta t)}\hat{T}_{\Delta t}\,\Omega(t)\,\mathrm{d}x\right)-\left(\oint_{S_0(t)} v_s(t)\,\mathrm{d}x+\iint_{G_0(t)}\Omega(t)\,\mathrm{d}x\right)$$
$$=\oint_{S_0(t)}\left(1-\frac{\alpha(s,t)}{2\pi}\right)q_{\Delta t}(z_s,t)\,\mathrm{d}s. \tag{5.36}$$

Relation (5.35) can be obtained in the same way as for the derivation of (5.20).

However, it cannot be considered as an independent condition on the solution for (5.15), as it depends on (5.20).

Summing relations (5.21) for all k and (5.36) and using

$$\iint_{R^2} \Omega(\boldsymbol{x}, t+\Delta t)\,\mathrm{d}x - \iint_{R^2} \Omega(\boldsymbol{x}, t)\,\mathrm{d}x = \oint_{S(t)} q_{\Delta t}(z_s, t)\,\mathrm{d}s$$

following (5.13), we obtain after transformations

$$\iint_{G(t+\Delta t)} \Omega(\boldsymbol{x}, t+\Delta t)\,\mathrm{d}x - \iint_{G(t)} \Omega(\boldsymbol{x}, t)\,\mathrm{d}x = \oint_{S(t+\Delta t)} \boldsymbol{v}_s(t+\Delta t)\,\mathrm{d}\boldsymbol{x} - \oint_{S(t)} \boldsymbol{v}_s(t)\,\mathrm{d}\boldsymbol{x} + o(\Delta t). \tag{5.37}$$

Thus, the variance of total vortex circulation in time is connected with the spins of bodies in the flow and is determined by the variance of circulation $\oint_{S(t)} \boldsymbol{v}(t)\,\mathrm{d}\boldsymbol{x}$ in time.

As a result, we obtain the generalization of the circulation conservation theorem (4.1) to the case of a flow with boundaries in the form

$$\iint_{G(t)} \Omega(\boldsymbol{x}, t)\,\mathrm{d}x - \oint_{S(t)} \boldsymbol{v}_s(t)\,\mathrm{d}\boldsymbol{x} = \text{const.}$$

Note that the variance of the integral (4.2) in time connects with forces influencing bodies placed in a flow (see Vladimirov 1977).

6. Conclusions

A complete theory of the vortex dynamics in two-dimensional flows of a viscous incompressible medium is given in this work based on the application of the product-integral method. As a result, the Cauchy problem is analytically solved for the quasi-linear Helmholtz equation in the form of product integrals in both unbounded and bounded flows under the no-slip boundary condition. The application of the product-integral method allows, by the most natural way, a generalization of the vortex dynamics concept in ideal fluids to the case of viscous flows, but the standard concepts in vortex theory must, however, be reconsidered. Elementary vortex objects are defined as two types of singular vortex filaments with equal but opposite intensities. The vorticity is considered as the macro-value being proportional to the concentration of elementary vortex filaments inhabiting the micro-level. The vortex motion of a viscous medium is represented as the stochastic motion of an infinite set of elementary vortex filaments on the micro-level governed by the stochastic differential equations, where the stochastic velocity component of each filament simulates the vorticity viscous diffusion, and the regular component is, however, the macro-value induced according to the Biot–Savart law and simulates the vorticity convective process. Also, the elementary vortex filaments are produced on flow boundaries to satisfy the no-slip condition.

This new approach and the analytic solution to the Helmholtz equation in the form of the product integral have obvious advantages. First, the product integration allows the calculation of the evolution of vorticity from its own initial field without any additional assumptions or other information about a flow. Secondly, the no-slip boundary condition is satisfied in a natural way and the laws of vortex production on boundaries are described exactly using the same technique of product integration. Note that there are considerable difficulties in other vortex methods in satisfying the no-slip condition. In particular, in Chorin's numerical method, the boundary-layer equations have to be solved to describe the vortex production on boundaries, which are not valid in some flow domains. The application of product integration includes the boundary of the flow in a natural way, and it is free from the shortcomings mentioned. Third, both analytic and approximate methods of the evaluation of product integrals are the subject of active investigations in contemporary mathematics. Also, there are effective numerical methods to evaluate product integrals directly which are not connected with applying the random walks technique. Therefore, the advanced theory of vortex dynamics in a viscous fluid flows allows the construction of effective and correct techniques to calculate flows of practical interest.

It should be noted that the method, developed in this work for incompressible viscous flows, can be generalized to the compressible viscous case. Indeed, the vortex dynamics of two-dimensional compressible viscous flows is described by an equation of the form (3.1) on the assumption that the kinematic viscosity ν is constant. However, the term $\nabla\cdot(\boldsymbol{u}\Omega)$ replaces $(\boldsymbol{u}\cdot\nabla)\,\Omega$ in the compressible Helmholtz equation, and there is a term of the form $(\nabla T\times\nabla S)_z$ from temperature–entropy inhomogenities of a flow. The main difference between compressible and incompressible flows consists in the fact that vorticity does not determine the velocity field uniquely. In the compressible case, the velocity has to be represented in the form $\boldsymbol{u}=\boldsymbol{u}_\Omega+\nabla\phi$, where $\boldsymbol{u}_\Omega$ is the solenoidal part of the velocity field which is determined by the vorticity through the Biot–Savart law, and $\nabla\phi$ is the potential velocity introduced by the compressibility. In this instance, additional information should be obtained about the compressible potential, ϕ, and the temperature–entropy inhomogenities of the flow to determine the vortex field evolution. It can only be obtained from a solution of the full set of gas dynamics equations. Nevertheless, assume that the potential ϕ, temperature and entropy are given. Then, the vortex model constructed in this work is generalized to the case of compressible media by adding the convective velocity of the form $\nabla\phi$ in (3.14) and introducing the volume production of elementary vortex filaments with the intensity $(\nabla T\times\nabla S)_z$ as macro-values. Then the vorticity evolution is determined by an expression of the form (5.11) in a small time step, where the surface integral is to be replaced by the volume integral of the same form with intensity $q(\boldsymbol{x},t)=(\nabla T\times\nabla S)_z+q_s\,\delta_s(\boldsymbol{x})$. As a result, we obtain an equation of the form (5.13) in which there will be terms of the form $\nabla\cdot(\boldsymbol{u}\Omega)$ and $(\nabla T\times\nabla S)_z$. This way, the vorticity evolution follows the Helmholtz equation in a compressible medium.

Developing some of the above ideas, we have constructed the vortex dynamics in three-dimensional viscous fluid flows, that we shall publish in Part 2 of this work.

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